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Remandedly

Groundely

Numerical work

Euclid Theorem

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COMPACTNESS

Bounded and closed subsets of Euclidean space have many interesting properties. Many of the important properties of $[0,1]$ follow from Heine-Borel theorem which states that every closed and bounded subset of real numbers is compact. The Heine-Borel theorem has many far reaching properties in analysis. Many of these guarantee that continuous functions defined on closed and bounded sets of real numbers are well behaved. e.g any such function is automatically bounded and uniform continuous. On the other hand function $f(x) = \frac{1}{x}$ defined on $(0,1)$ is neither bounded nor uniformly continuous

Boundness can not be satisfactorily defined in general topological spaces

In Euclidean space, boundedness and closedness can be described in several ways e.g by

(i) Bolzano-Weierstrass Theorem or

(ii) Heine-Borel theorem or

(iii) Cantor's theorem on nested closed sets.

These theorems are equivalent in a Euclidean space, but not so in an arbitrary top-space. These can be used to define compactness in a general top-space. So we have different types of compactness.

Different Types of Compactness

A top-space is said to be

(i) Compact if every open covering of X has a finite sub-covering

(ii) Countably compact if every countable open covering of X has a finite sub-covering

(iii) Sequentially compact if every infinite sequence in X

Contains a convergent subsequence

(iv) Frechet Compact (B.W Compact) if every infinite subset of X has an accumulation point.

These types of compactness may also be defined for the subsets of X with respect to their subspace topologies.

Lindeloff may be considered as a kind of Compactness. The four types of compactness and Lindeloff property are not equivalent but they are not independent of each other.

Other types of compactness viz. meta & para Compactness are defined in terms of refinement of coverings. All the types of Compactness are more appropriately described in terms of nets, filters

Of the four types of Compactness, sequential Compactness and B.W compactness are considered to be less important. Also properties of Countably Compactness are almost similar to the corresponding properties of Compact space, with the condition of Countability.

Cover

A covering or cover of a set E is a collection of set $\{A_\alpha : \alpha \in I\}$ such that

$$E \subseteq \bigcup_\alpha A_\alpha.$$

If all the sets of a covering are open sets, it is called an open covering

If some finite subcollection of a covering of E is also a covering of E, we say that the covering is reducible to a finite subcovering

Compact Top-Space

A top-space X is said to be Compact if every open covering for X has a finite

subcovering.

A subset A of a top-space X is compact if every covering of A by open sets of X has a finite subcovering

A subspace Y of X is said to be compact if Y with relative topology, is compact.

Theorem # If E is a subset of a subspace (Y, T_Y) of a top-space (X, T), Then E is T_Y-Compact iff it is T-Compact

Proof # Let E be T_Y-compact and $\{G_\alpha : \alpha \in I\}$ is some T-open covering of E i.e

$$E \subseteq \bigcup_{\alpha \in I} G_\alpha$$

$$\Rightarrow \quad X \cap E \subseteq X \cap \left(\bigcup_{\alpha} G_\alpha\right)$$

$$\Rightarrow \quad E \subseteq \bigcup_{\alpha \in I} (X \cap G_\alpha) \qquad \text{Distributive Law}$$

So The family of sets $\{X \cap G_\alpha : \alpha \in I\}$ forms a T_Y-open covering of E.

$\because$ E is T_Y-compact

$\therefore$ there exists a finite subcovering $\{X \cap G_i : i = 1, 2 \ldots n\}$ of E

Now

$$E \subseteq \bigcup_{i=1}^{n} (X \cap G_i) \subseteq \bigcup_{i=1}^{n} G_i \qquad \because X \cap G_i \subseteq G_i$$

Thus $\{G_i : i = 1, 2 \cdots n\}$ is a finite subcovering of the covering $\{G_\alpha : \alpha \in I\}$ of E. Hence E is T-compact.

Now suppose that E is T-compact and $\{H_\alpha : \alpha \in I\}$ is some T_Y-open covering of E.

from the definition of the induced topology, each $H_\alpha = X \cap G_\alpha$ for some T-open set G_α

Now $E \subseteq \bigcup_{\alpha \in I} H_\alpha = \bigcup_{\alpha \in I} (X \cap G_\alpha)$

$= X \cap (\bigcup_{\alpha \in I} G_\alpha) \subseteq \bigcup_{\alpha \in I} G_\alpha$

$\Rightarrow$ The family $\{G_\alpha\}$ is a T- open covering of E

$\because$ E is T- compact

$\therefore$ there is finite subcovering $\{G_i : i = 1, 2 \ldots n\}$ of $\{G_\alpha\}$ such that

$$E \subseteq \bigcup_{i=1}^{n} G_i$$

$$\Rightarrow \quad X \cap E \subseteq X \cap (\bigcup_{i=1}^{n} G_i)$$

$$E \subseteq \bigcup_{i=1}^{n} (X \cap G_i) = \bigcup_{i=1}^{n} H_i$$

Thus $\{H_i : i = 1, 2 \ldots n\}$ is a finite subcovering of E from $\{H_\alpha : \alpha \in I\}$

Hence E is T_y – compact

Comments # The above theorem shows that compactness, is an absolute property

Example # Any top- space (X, T) in which T is finite or T consists of a finite no of elements is compact because in this case every open cover of X is finite.

Example # Let X be an infinite set with a co-finite topology T, then (X, T) is compact.

Sol Let $C = \{U_\alpha : \alpha \in I\}$ be an open cover of X. let $U_\beta \in C$. Then

U_β' is finite set

let $U_\beta' = \{x_1, x_2 \ldots x_n\}$

$\because$ C is cover of X

$\therefore$ every point of X is in some U_α.

$\Rightarrow$ There are open sets $U_{\alpha_i} \in C$ such that

$x_i \in U_{\alpha_i}$ $\quad i = 1, 2 \ldots n$

Now $U_\beta' = X - U_\beta = \{x_1, x_2, \ldots x_n\}$

$\Rightarrow$ U_β contains infinite points of X other than $x_1, x_2 \ldots x_n$ i.e U_β covers an infinite part of X

Thus the finite sub-collection

$\{U_\beta, U_{\alpha_1}, U_{\alpha_2} \ldots U_{\alpha_n}\}$ is finite open cover of X. Hence (X, T) is compact

Example # The real line R is not Compact.

Consider the collection

$C = \{G_n : n \in N\}$ where $G_n = (-n, n)$

clearly $\bigcup_{n \in N} G_n = R$ i.e C is an open cover.

Suppose that this open cover has a finite sub-cover $\{G_{n_1}, G_{n_2}, \ldots G_{n_k}\}$ i.e

$$R = \bigcup_{i=1}^{k} G_{n_i}$$

Let $m = \max(n_1, n_2 \ldots n_k)$

Then $G_{n_i} \subseteq G_m \quad i = 1, 2, \ldots k$

So that $\bigcup_{i=1}^{k} G_{n_i} \subseteq G_m$

$\Rightarrow R \subseteq G_m \subseteq R$

$\Rightarrow R = (-m, m) = G_m$

Which is impossible because for any $x \in R$, $|x| > m$ $x \notin G_m$. So C has no finite sub-cover of R. Hence R is not compact.

Example # An open interval in the real number space (R, T) may not be compact. e.g

Consider the open interval $(0, 1)$

Collection $C = \{G_n = (\frac{1}{n}, 1 - \frac{1}{n}) : n = 3, 4 \ldots\}$ is an open cover for $(0, 1)$

Now consider a finite subcollection of C say $\{G_i : i = 3, \ldots k\}$

Then $\bigcup_{i=3}^{k} G_i = (1/3, 2/3) \cup (1/4, 3/4) \cup \cdots \cup (1/k, 1-1/k)$

does not contain $1/k$ and $1/k \in (0, 1)$

If $k = 7$,

$$\bigcup_{i=3}^{7} G_i G = (1/3, 2/3) \cup (1/4, 3/4) \cup (1/5, 4/5) \cup \cdots \cup (1/7, 6/7)$$

$$1/7 \notin \bigcup_{i=3}^{7} G_i \quad \text{and} \quad 1/7 \in (0, 1)$$

Thus $(0, 1)$ is not compact

Similarly the open covering $\{(1/n, n/n+1) : n = 2, 3 \ldots\}$ of the open interval $(0, 1)$ has no finite subcovering

Example # Let X be an infinite set with discrete topology. Then X is not compact space

Sol # Since topology on X is discrete, so every singleton subset of X is also open

Let $C = \{\{x\} : x \in X\}$ be an open cover of X. no finite subcollection will be an open cover of X because union of finite subcollection will be finite and so will not be equal to X. Hence X is not a compact space

Theorem # (Heine Borel theorem)

Every closed interval in the real number space (X, T), with the usual topology, is compact

Proof # Let $[a, b]$, with $a < b$ be a closed interval in R and C be an open cover of $[a, b]$ in (R, T). Let P be collection of all those elements $x \in [a, b]$, such that there exists a finite sub-covering of $[a, x]$. Then $a \in P$. Also b is an upper bound of the set P. By least upper bound property of real numbers P has a lub

Let c be the least upper bound of P. As a is contained in some member of C, which must contain points in (a b] Also (since an open set is union of some open intervals and any open interval, containg a can not have a as its right endpoint. It follows that

$$a < c$$

Let U be a member of C such that $c \in U$. Let $d \in (a\ c) \cap U$. Then [a d) has an open covering by a finite sub-collection C' of C. Then $C' \cup U$ forms a finite subcovering of C for [a c]. If $c \neq b$, then there exist a number e, such that $e \in U$ and $(c, e) \subseteq U$ and in that case $C' \cup U$ forms a finite subcovering of C for [a, e], so that $e \in P$, which is not possible because $e > c$ (and c is l.u.b of P). Thus $c = b$ and therefore $C' \cup U$ forms a finite subcovering of C for [a b]. Hence closed interval [a b] is compact in R

Finite Intersection Property # (F.I.P)

A collection $\{A_i : i \in I\}$ of subsets of a given set X is said to have finite intersection property, if every finite sub-collection of $\{A_i : i \in I\}$ has non-empty intersection.

Theorem # A top-space (X T) is compact iff any family of closed sets in (X.T) having the finite intersection property has a non-empty intersection.

Proof # Let X be compact and $\{F_i : i \in I\}$ be a class of closed sets with F.I.P. We are to prove that

$$\bigcap_{i \in I} F_i \neq \emptyset$$

Let on the contrary $\bigcap_{i \in I} F_i = \emptyset$

Complementing on both sides, we have

$$\left(\bigcap_{i\in I} F_i\right)' = X$$

$$\bigcup_{i\in I} F_i' = X \qquad \text{(De Morgan law)}$$

where F_i' is open for each i

Thus $\{F_i' : i \in I\}$ is an open cover of X

$\because$ X is compact

$\therefore$ This open cover has a finite sub-cover $\{F_j' : j = 1, 2 \cdots n\}$

i.e

$$X = \bigcup_{j=1}^{n} F_j'$$

Complementing on both sides

$$\phi = \left(\bigcup_{j=1}^{n} F_j'\right)'$$

$$= \bigcap_{j=1}^{n} F_j \qquad \text{De Morgan law}$$

$\Rightarrow$ finite subcollection $\{F_j : j = 1, 2 \cdots n\}$ of $\{F_i : i \in I\}$ has an empty intersection. But this is a contradiction to F.I.P. Hence

$$\bigcap_{i\in I} F_i \neq \phi$$

Conversely suppose that every class of closed sets with F.I.P has non-empty intersection.

Let $\{G_\alpha : \alpha \in I\}$ be an open cover of X i.e

$$\bigcup_{\alpha\in I} G_\alpha = X$$

Complementing on both sides, we have.

$$\bigcap_{\alpha\in I} G_\alpha' = \phi$$

$\{G'_\alpha : \alpha \in I\}$ forms a family of closed sets with empty intersection. It follows from the given condition, that the family $\{G'_\alpha : \alpha \in I\}$ can not possess the F.I.P. Hence a certain finite sub collection say $\{G'_i : i = 1, 2, \ldots n\}$ must have have an empty intersection i.e

$$\bigcap_{i=1}^{n} G'_i = \emptyset$$

Complementing

$$\bigcup_{i=1}^{n} G_i = X$$

$\Rightarrow \{G_i : i = 1, 2 \cdots n\}$ forms a finite sub-covering of $\{G_\alpha : \alpha \in I\}$ for X. Hence (X, T) is a compact.

OR

We can prove the converse of above theorem as

Let $\{G_\alpha : \alpha \in I\}$ be an open covering of X. i.e

$$\bigcup_{\alpha \in I} G_\alpha = X$$

Suppose that this cover has no finite subcovering i.e. for every subset I' of I we have

$$\bigcup_{\alpha \in I'} G_\alpha \neq X$$

$$\Rightarrow \quad \bigcap_{\alpha \in I'} G'_\alpha \neq \emptyset \qquad \forall\, I' \subseteq I$$

Now $\{G'_\alpha : \alpha \in I\}$ is class of closed sets with F.I.P Hence by given condition.

$$\bigcap_{\alpha \in I} G'_\alpha \neq \emptyset$$

Complementing

$$\bigcup_{\alpha \in I} G_\alpha \neq X$$

a contradiction. Hence $\{G_\alpha : \alpha \in I\}$ has a finite subcover. Thus X is compact

Comments # This Theorem is a sort of dual to definition of compactness.

Theorem # A top-space is compact $\Leftrightarrow$ every class of closed sets with empty intersection has a finite subclass with empty intersection.

Proof # Let X be a compact space and $\{F_i : i \in I\}$ be the collection of closed sets with empty intersection i.e

$$\bigcap_{i \in I} F_i = \emptyset$$

We have to prove that $\{F_i : i \in I\}$ contains a finite subclass with empty intersection.

$$\therefore \quad \bigcap_{i \in I} F_i = \emptyset$$

Complementing

$$\bigcup_{i \in I} F_i' = \emptyset' = X$$

where each F_i' is open.

$\Rightarrow \{F_i' : i \in I\}$ is an open cover for X. Since X is compact, this open cover will have finite subcover of X

Let $\{F_j' : j \in I\} = \{F_j' : j = 1, 2 \ldots n\}$ be the finite subcover of X i.e

$$\bigcup_{j=1}^{n} F_j' = X$$

Complementing

$$\bigcap_{j=1}^{n} F_j = \emptyset$$

$\Rightarrow \{F_i : i \in I\}$ Contains a finite sub-class with empty intersection.

Conversely suppose that every class of closed sets with empty intersection has a finite sub-class with empty intersection.

We have to prove that X is compact.

Let $\{G_\alpha : \alpha \in I\}$ be an open cover for X. Then

$$X = \bigcup_{\alpha \in I} G_\alpha$$

Complementing

$$X^c = \left(\bigcup_{\alpha \in I} G_\alpha\right)^c$$

$$\phi = \bigcap_{\alpha \in I} G_\alpha^c$$

$\Rightarrow \{G_\alpha^c : \alpha \in I\}$ is a class of closed sets with empty intersection. So by condition there is a finite subclass $\{G_i : i = 1, 2, \cdots n\}$ with empty intersection i.e

$$\bigcap_{i=1}^{n} G_i' = \phi$$

Complementing $\bigcup_{i=1}^{n} (G_i')' = X \Rightarrow \bigcup_{i=1}^{n} G_i = X$

$\Rightarrow \{G_\alpha : \alpha \in I\}$ has a finite subcover

$\Rightarrow X$ is a compact space.

Theorem # Any continuous image of a compact space is compact.

Proof # Let $f : X \longrightarrow Y$ be a continuous mapping of a compact space X into an arbitrary topological space Y. We are to show that $f(X)$ is a compact subspace of Y. Let τ, τ' be topologies on X, Y and $\{G_i : i \in I\}$ be an open cover of $f(X)$. Then each G_i is intersection of $f(X)$ with some open set of Y.

Let $\{H_i : i \in I\}$ be collection of open sets in Y such that

$$G_i = f(X) \cap H_i \quad \forall i \in I$$

$\because \{G_i : i \in I\}$ is an open cover of $f(X)$

$$\therefore \bigcup_{i \in I} G_i = f(X)$$

Now $\bar{f}^{-1}(G_i) = \bar{f}^{-1}[f(X) \cap H_i]$

$$= \bar{f}^{-1}(f(X)) \cap \bar{f}^{-1}(H_i)$$

$$= X \cap \bar{f}^{-1}(H_i)$$

$$= \bar{f}^{-1}(H_i) \quad \because \bar{f}^{-1}(H_i) \subseteq X$$

$$\bar{f}^{-1}(G_i) = \bar{f}^{-1}(H_i)$$

$$\bigcup_{i \in I} \bar{f}^{-1}(G_i) = \bigcup_{i \in I} \bar{f}^{-1}(H_i)$$

$$\bar{f}^{-1}\left[\bigcup_{i \in I} G_i\right] = \bigcup_{i \in I} \bar{f}^{-1}(H_i)$$

$$\bar{f}^{-1}(f(X)) = \bigcup_{i \in I} \bar{f}^{-1}(H_i)$$

$$X = \bigcup_{i \in I} \bar{f}^{-1}(H_i)$$

$\Rightarrow \{\bar{f}^{-1}(H_i) : i \in I\}$ is an open covering of X

$\because X$ is compact

$\therefore$ This open covering has a finite subcovering $\{\bar{f}^{-1}(H_j) : j = 1, 2, \ldots n\}$ i.e

$$X = \bigcup_{j=1}^{n} \bar{f}^{-1}(H_j)$$

$$= \bar{f}^{-1}\left(\bigcup_{j=1}^{n} H_j\right)$$

$$\Rightarrow f(X) = \bigcup_{i=} H_j$$

$$f(X) \cap f(X) = \left(\bigcup_{j=1}^{n} H_j\right) \cap f(X)$$

$$f(X) = \bigcup_{j=1}^{n} (H_j \cap f(X))$$

$$= \bigcup_{j=1}^{n} G_j \qquad G_j = H_j \cap f(X) \in \{G_i : i \in I\}$$

$\Rightarrow \{G_j : j = 1, 2 \ldots n\}$ is a finite sub cover of $\{G_i : i \in I\}$ for $f(X)$.

Hence $f(X)$ is compact.

Theorem# The image of a compact set under a continuous mapping is compact.

Proof# Let $f: X \longrightarrow Y$ be a continuous fn from a top-space $(X, \mathcal{T}_1)$ into $(Y, \mathcal{T}_2)$ and A be a compact subset of X. We want to prove that

$B = f(A)$ is compact subset of Y

Now $f(A)$ will be compact if it is compact as a subspace of Y. In view of the definition of the topology of a subspace, this is equivalent to saying that $f(A)$ is compact if every open covering of $f(A)$ in X has a finite sub covering.

Let $\{G_\alpha : \alpha \in I\}$ be an open cover of $f(A) = B$ in Y i.e

$$f(A) \subseteq \bigcup_{\alpha \in I} G_\alpha$$

$\because$ f is continuous

$\therefore$ each $f^{-1}(G_\alpha)$ is open in X

Now

$$f(A) \subseteq \bigcup_{\alpha \in I} G_\alpha$$

$$\Rightarrow A \subseteq f^{-1}\left(\bigcup_{\alpha \in I} G_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(G_\alpha)$$

$\Rightarrow$ The family $\{f^{-1}(G_\alpha)\}$ is an open cover for the subset A

Since A is compact in X, this cover has a finite subcover $\{f^{-1}(G_i) : i = 1, 2, \ldots n\}$

Therefore $$A \subseteq \bigcup_{i=1}^{n} f^{-1}(G_i) = f^{-1}\left(\bigcup_{i=1}^{n} G_i\right)$$

$$\Rightarrow \quad f(A) \subseteq \bigcup_{i=1}^{n} G_i$$

$\Rightarrow$ Sub-family $\{G_i : i = 1, 2 \ldots n\}$ is a a subcover for $f(A)$. Hence $f(A)$ is Compact.

Subspaces of a Compact Spaces

Theorem #

(a) # A subspace of a compact space need not be compact

(b) # Any closed subspace of a compact space is Compact

(c) # The union of a finite collection of Compact subspaces of a topological space is compact

Proof # (a) $[0\ 1]$ being closed interval in the real space (R, T) is compact and $(0, 1)$ being subspace of $[0\ 1]$ is not compact. Hence subspace of a compact space need not Compact

(b) # Let $(X\ T)$ be a compact space and A be a closed subspace of $(X\ T)$. Let $\{F_\alpha : \alpha \in I\}$ be a family of closed sets having F.I.P. Then each F_α is intersection of A with some closed set of X. Since A is closed, each F_α is also closed in $(X.T)$. Thus $\{F_\alpha : \alpha \in I\}$ is also a family of closed sets in $(X\ T)$, having finite intersection property. As $(X\ T)$ is compact, the family

has non-empty intersection i.e $\cap\{F_\alpha : \alpha \in I\} \neq \emptyset$ and so $(A\ T_A)$ is compact

II- Method

Let (X, T) be compact and A be a closed subspace of X. Let $C = \{G_\alpha : \alpha \in I\}$ be an open covering for A i.e

$$A \subseteq \bigcup_{\alpha \in I} G_\alpha \qquad A \cup A^c = X$$

$$A^c \cup A \subseteq A^c \cup \left(\bigcup_{\alpha \in I} G_\alpha\right)$$

$$X \subseteq A^c \cup \left(\bigcup_{\alpha \in I} G_\alpha\right) \subseteq X$$

$$\Rightarrow X = A^c \cup \left(\bigcup_{\alpha} G_\alpha\right)$$

$\Rightarrow \{A^c, G_\alpha : \alpha \in I\}$ is a an open covering for X

$\because$ X is compact

$\therefore$ $\{A^c, G_\alpha : \alpha \in I\}$ has a finite subcovering $\{A^c, G_1, G_2, \ldots G_n\}$ of X i.e

$$X = A^c \cup G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_n$$

Complementing

$$\emptyset = A \cap (G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_n)^c$$

$$= A \cap \left(\bigcup_{i=1}^{n} G_i\right)^c$$

$$\Rightarrow A \subseteq \left[\left(\bigcup_{i=1}^{n} G_i\right)^c\right]^c = \bigcup_{i=1}^{n} G_i$$

$\Rightarrow$ sub-family $\{G_i : i = 1, 2 \ldots n\}$ is a finite subcover of A from cover $\{G_\alpha : \alpha \in I\}$. Hence A is compact.

III- Method (Direct Method)

Let y be a closed subspace of a compact space $(X\ T)$ and $\{G_\alpha : \alpha \in I\}$ be an open cover of y. Since y is a subspace of X, each G_α is open

Relative to Relative topology T_Y (induced topology).
Therefore there is a family $\{H_\alpha : \alpha \in I\}$ of open sets in X such that

$$G_\alpha = Y \cap H_\alpha \qquad H_\alpha \in T, \ \alpha \in I$$

Since Y is closed, Y^c is open

Also

$$Y \cup Y^c = X$$

$\because \{G_\alpha : \alpha \in I\}$ is an open covering of Y

$$\therefore \quad Y \subseteq \bigcup_{\alpha \in I} G_\alpha = \bigcup_{\alpha \in I} (Y \cap H_\alpha)$$

$$= Y \cap \left(\bigcup_{\alpha \in I} H_\alpha\right)$$

$$\Rightarrow \quad Y^c \cup Y \subseteq Y^c \cup \left(Y \cap \left(\bigcup_{\alpha \in I} H_\alpha\right)\right)$$

$$= (Y^c \cup Y) \cap \left(Y^c \cup \left(\bigcup_{\alpha} H_\alpha\right)\right)$$

$$= X \cap \left(Y^c \cup \left(\bigcup_{\alpha} H_\alpha\right)\right)$$

$$= Y^c \cup \left(\bigcup_{\alpha} H_\alpha\right)$$

$$\Rightarrow \quad X \subseteq Y^c \cup \left(\bigcup_{\alpha} H_\alpha\right) \subseteq X$$

$$\Rightarrow X = Y^c \cup \left(\bigcup_{\alpha} H_\alpha\right)$$

$\Rightarrow \{\{H_\alpha : \alpha \in I\}, Y^c\}$ is an open covering for X

$\because$ X is compact

$\therefore$ This covering has a finite subcovering $\{H_i : i = 1, 2 \cdots n, Y^c\}$ i.e

$$X = Y^c \cup \left(\bigcup_{i=1}^{n} H_i\right)$$

$$Y = Y \cap X$$

$$= Y \cap \left(Y^c \cup \left(\bigcup_{i=1}^{n} H_i\right)\right)$$

$$= (Y \cap Y^c) \cup \left(Y \cap \left(\bigcup_{i=1}^{n} H_i\right)\right)$$

$$= \emptyset \cup \left(\bigcup_{i=1}^{n} H_i \cap Y\right) = \bigcup_{i=1}^{n} (H_i \cap Y)$$

$$Y = \bigcup_{i=1}^{n} G_i \qquad G_i = H_i \cap Y \in \{G_\alpha, \alpha \in I\}$$

Hence Y is Compact

(C) The union of a finite collection of compact subspaces of a topological space is compact.

Let $Y_1, Y_2 ----, Y_n$ be a finite collection of compact spaces with their relative topologies.
Let $Y = Y_1 \cup Y_2 \cup Y_2 \cup Y_3 \cup ---- \cup Y_n$.
Let $C = \{G_\alpha : \alpha \in I\}$ be an open covering of Y. Then

$$Y \subseteq \bigcup_{\alpha \in I} G_\alpha.$$

$\because Y_i \subseteq Y \qquad \forall i$
$\therefore Y_i \subseteq \bigcup_{\alpha \in I} G_\alpha \qquad \forall i$

$\Rightarrow \{G_\alpha : \alpha \in I\}$ is an open covering of each of Y_i. Since each Y_i is compact, therefore there exists an open cover of $\{G_\alpha : \alpha \in I\}$ for each Y_i.

Let $C_1 = \{G_{1,1}, G_{1,2}, G_{1,3}, ----- G_{1,r_1}\}$ a finite subcover for Y_1
$C_2 = \{G_{2,1}, G_{2,2}, G_{2,3} ------ G_{2,r_2}\}$ " " " Y_2
$C_3 = \{G_{3,1}, G_{3,2}, ---- G_{3,r_3}\}$ " ... Y_3
$\vdots$
$C_n = \{G_{n,1}, G_{n,2} ---- G_{n,r_n}\}$ " " " ... Y_n

$\Rightarrow Y_1 \subseteq G_{1,1} \cup G_{1,2} \cup G_{1,3} \cup ----- \cup G_{1,r_1}$
$Y_2 \subseteq G_{2,1} \cup G_{2,2} \cup G_{2,3} \cup ----- \cup G_{2,r_2}$
$\vdots$
$Y_n \subseteq G_{n,1} \cup G_{n,2} \cup G_{n,3} \cup ---- \cup G_{n,r_n}$

$Y = Y_1 \cup Y_2 \cup Y_2 \cup \cdots \cup Y_n \subseteq G_{1,1} \cup G_{1,2} \cup \cdots \cup G_{1,k_1} \cup G_{2,1} \cup G_{2,2} \cup \cdots \cup G_{2,k_2} \cdots \cdots \cup G_{n,1} \cup G_{n,2} \cup \cdots \cup G_{n,k_n}$

$$Y \subseteq \left(\bigcup_{i=1}^{k_1} G_{1,i}\right) \cup \left(\bigcup_{i=1}^{k_2} G_{2,i}\right) \cup \cdots \cdots \cup \left(\bigcup_{i=1}^{k_n} G_{n,i}\right)$$

$\Rightarrow$ sets of $C_1, C_2, C_3 \cdots C_n$ form a finite subcover for Y. Hence Y is compact

Basic Open Cover

Let (X, T) be a topological space. Then an open cover $\{B_\alpha : \alpha \in I\}$, where B_α is a basic open set, is called basic open cover for X

OR

An open cover whose each element is basic open set is called basic open cover

Sub-Basic Open Cover

An open cover of a top-space (X, T) is sub-basic open cover if its each element is from sub-base

Theorem # (Alexander's subbase theorem).

Let S be a sub-base of the topology T of a topological space (X, T). If, for every covering of X by a collection of members of S, there exists a finite sub-covering, then (X, T) is a compact space. OR

A top-space is compact if every sub-basic open cover has a finite sub-cover

Proof

Theorem# A top- space is compact iff every basic open cover has a finite subcover

Proof# Suppose that X is compact and let $\{B_\alpha : \alpha \in I\}$ be a basic open cover for X. Then obviously $\{B_\alpha : \alpha \in I\}$ is an open cover for X. Since X is compact, this open cover has a finite subcover of X.

Conversely let every basic open cover has a finite sub-cover. Let $\beta = \{B_\alpha : \alpha \in I\}$ be an open base for X. We are to prove that X is compact

Let $\{G_i : i \in I\}$ be an open cover of X i.e

$$X = \bigcup_{i \in I} G_i$$

Therefore each $x \in X$, is contained in G_i for some $i \in I$. Since β is a base for X, each element of an open set G_i is contained in some B_α and B_α is contained in G_i i.e

$$x \in B_\alpha \subseteq G_i$$

Taking Union over all the elements of x

$$\bigcup_{x \in X} \{x\} \subseteq \bigcup_{\alpha \in I} B_\alpha \subseteq \bigcup_{i \in I} G_i$$

$$\Rightarrow X \subseteq \bigcup_{\alpha \in I} B_\alpha \subseteq X$$

$$\Rightarrow X = \bigcup_{\alpha \in I} B_\alpha$$

$\Rightarrow \{B_\alpha : \alpha \in I\}$ is a basic open cover of X.

By given hypothesis this basic open cover has a finite sub-cover say $\{B_j : j = 1, 2 \dots n\}$ i.e

$$X = \bigcup_{j=1}^{n} B_j$$

But each B_j is contained in some G_j from $\{G_i : i \in I\}$ i.e

$$X = \bigcup_{j=1}^{n} B_j \subseteq \bigcup_{j=1}^{n} G_j \subseteq X$$

$$\Rightarrow X = \bigcup_{j=1}^{n} G_j$$

which shows that open cover for X has a finite sub-cover. Hence X is compact.

Nested Interval Property

Let $\{I_n : n \in N\}$ be a sequence of nested closed intervals i.e

$$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \cdots$$

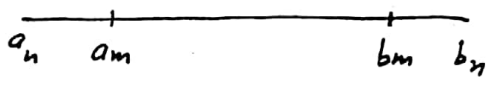
Then there exist at least one point common to all of them i.e

$$\bigcap_{n \in N} I_n \neq \phi$$

Proof # We claim that

$$a_m \leq b_n \quad \forall\, m, n \in N$$

Case I Suppose that $m > n$, then in the interval $I_m = [a_m, b_m]$, we have $a_m < b_m$



Since the sequence is nested, therefore

$$a_m < b_n$$

Case-II Suppose that $m \leq n$

Since the sequence is nested, so $a_m < a_n$ holds but $a_n < b_n$

$$\Rightarrow a_m < b_n$$

We consider

$A = \{a_1, a_2 \cdots a_n, \cdots\}$ the set end points of $\{I_n : n \in N\}$

Suppose

$$\sup(A) = p < b_n \quad \forall n \in N$$

Also

$$a_n < p \quad \forall n$$

$$a_n < p < b_n$$

$$\Rightarrow p \in I_n = [a_n, b_n] \quad \forall n \in N$$

$$\Rightarrow \bigcap_{n \in N} I_n \neq \phi$$

Length of Interval

Let $I_n = [a_n, b_n]$ be a closed / open interval of R, then length of I_n is denoted by $|I_n|$ and is defined as

$$|I_n| = (b_n - a_n)$$

When $n \to \infty$, we can take limiting value.

$$\lim_{n\to\infty} |I_n| = 0$$

By definition of limit this means that for every $\epsilon > 0$, there is a natural no n_0 such that

$$|(b_n - a_n) - 0| < \epsilon \qquad \forall\, n \geq n_0$$

Problem # Show that in a sequence of closed nested intervals, the common point is unique.

Sol # Suppose that the common point is not unique but p_1, p_2 are two distinct common points. Then

$$p_1, p_2 \in I_n \qquad \forall n \in N$$

Now

$$\lim_{n\to\infty} |I_n| = 0$$

$\Rightarrow$ for given $\epsilon > 0$ there exists a natural number n_0 such that

$n \geq n_0 \Rightarrow |b_n - a_n| < \epsilon$

Consider $I_{n_0} = [a_{n_0} \quad b_{n_0}]$

$$|I_{n_0}| = |a_{n_0} - b_{n_0}|$$

If $p_1 \neq p_2$, then $|p_1 - p_2| \neq 0$

Let $|p_1 - p_2| = \delta > 0$

Taking $|I_{n_0}| < \delta = \epsilon$

$\Rightarrow$ both of p_1 & $p_2 \notin I_{n_0}$ because $|p_1 - p_2| = \delta$

Hence the common point is unique.

Theorem # Every closed and bounded interval in R is compact. OR

Let $I_1 = [c_1, d_1]$ be a closed interval in R and C be an open cover of I_1, then there exists a finite sub-cover of I_1.

Proof # Let $I_1 = [c_1, d_1]$ be closed and bounded subset of R

Let $C = \{(a_\alpha, b_\alpha) : \alpha \in I\}$ be an open cover of I_1.

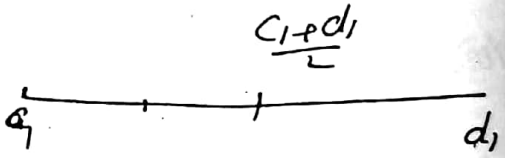
We are to show C has finite subcover or I_1 is compact.

Let I_1 be not compact. There does not exist finite subcover of C which covers I_1.

Bisecting I_1, we get two intervals

$[c_2, d_2] = \left[c_1 \quad \frac{c_1+d_1}{2}\right]$

$[c_2', d_2'] = \left[\frac{c_1+d_1}{2} \quad d_1\right]$



Then one of these do not possess a finite subcover because otherwise I_1 is compact.

Let $I_2 = [c_2, d_2]$ does not have finite subcover.

Bisecting $[c_2, d_2]$, we get two intervals

$[c_3 \quad d_3] = \left[c_2 \quad \frac{c_2+d_2}{2}\right]$

$[c_3' \quad d_3'] = \left[\frac{c_2+d_2}{2} \quad d_2\right]$

Again one of these has no finite subcover say

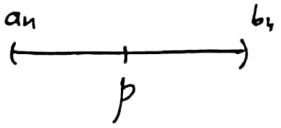
$$I_3 = [c_3 \quad d_3]$$

Continuing this process of bisecting and selecting the interval which donot have finite subcover, we construct nested sequence of closed intervals such that

$$\lim_{n\to\infty} |I_n| = 0$$

By nested interval property $\bigcap_{n\in N} I_n \neq \emptyset$ and there exist a unique point common to all of them. Suppose p is the common point and consider $p \in (a_n \; b_n) \in C$.

Let $\min\{|p-a_n|, |b_n-p|\} = \delta$



Since $\lim_{n\to\infty} |I_n| = 0$

so for $\epsilon > 0$ $\exists\, n_0$ such that

$$|I_n| < \epsilon \qquad \forall n \geq n_0$$

Let $\epsilon = \delta$, then $\exists$ a natural number n_0 such that

$$|I_n| < \delta \qquad \forall n \geq n_0$$

$\Rightarrow$ Either (a_n, p) . (p, b_n) covers I_n, a contradiction, because each $I_n, n\in N$, does not possess a finite sub-cover.

Hence I_1 is compact.

Theorem# Let X, Y, Z be top-spaces such that

$$Z \subseteq Y \subseteq X$$

Then Z is compact relative to Y iff Z is compact relative to X

Proof# Let Z be compact relative to X

and let $\{G_i : i \in I\}$ be a class of sets open in Y such that

$$Z \subseteq \bigcup_{i \in I} G_i$$

Then $G_i = Y \cap H_i \quad \forall i \quad H_i$ open in X

Clearly $\{H_i : i \in I\}$ is an open cover of Z in X. Since Z is compact in X, therefore this cover has a finite subcover $\{H_j : j = 1, 2 \ldots n\}$ i.e

$$Z \subseteq \bigcup_{j=1}^{n} H_j$$

$$Z \cap Z \subseteq Y \cap \bigcup_{j=1}^{n} H_j \qquad \because Z \subseteq Y$$

$$Z \subseteq \bigcup_{j=1}^{n} (Y \cap H_j)$$

$$Z \subseteq \bigcup_{j=1}^{n} G_j \qquad G_j = Y \cap H_j \quad j = 1, 2 \ldots n.$$

which shows that Z is compact relative to Y.

Conversely let Z be compact relative to Y and Let $\{H_\alpha : \alpha \in I\}$ be class of open sets of X which covers Z i.e

$$Z \subseteq \bigcup_{\alpha \in I} H_\alpha$$

Let $G_\alpha = Y \cap H_\alpha$ where G_α is open relative to Y

$\because$ Z is compact relative to Y

$$\therefore \quad Z \subseteq \bigcup_{i=1}^{n} G_i = Y \cap \bigcup_{i=1}^{m} H_i$$

$$\Rightarrow Z \subseteq \bigcup_{i=1}^{n} H_i$$

$\Rightarrow Z$ is compact relative to X

<u>Exercise</u> # A compact subset of a compact space may not be closed.

<u>Sol</u> # Let $X = \{a, b, c\}$

$\mathcal{F} = \{ \phi \ \{a\}, \{b\} \ \{a,b\}, X \}$

$A = \{a,b\}$

$A' = c \notin \mathcal{F}$

$\Rightarrow$ A is not closed.

But A is and X is compact

Exercise # The union of two compact sets is compact

Sol # Let $(X, \mathcal{F})$ be a top-space and A_1, A_2 be Compact subset of X. If $B = A_1 \cup A_2$. We want to prove that B is also compact. Let $\{G_\alpha\}_{\alpha \in I}$ be an open cover for B. Because.

$A_1 \subseteq B \quad A_2 \subseteq B$, $\{G_\alpha : \alpha \in I\}$ is an open cover for both A_1 & A_2

Since A_1, A_2 are Compact, there are finite subcover $\{G_i\}_{i=1}^{n}$ and $\{G_\beta\}_{\beta=1}^{n}$ resp. for A_1 and A_2 i.e

$$A_1 \subseteq \{G_i\}_{i=1}^{n}$$

$$A_2 \subseteq \{G_\beta\}_{\beta=1}^{n}$$

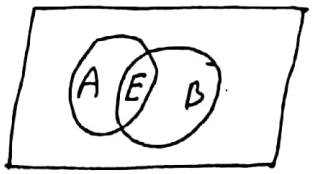
$$\Rightarrow A_1 \cup A_2 \subseteq \{G_i\}_{i=1}^{n} \cup \{G_\beta\}_{\beta=1}^{n}$$

Hence every open cover for B has a finite subcover and so it is compact.

Exercise # Let $(X, \mathcal{F})$ be a top-space. $A \subseteq X$ & $B \subseteq X$. If A is Compact and B is closed., Then $A \cap B$ is Compact

Sol Let $E = A \cap B$

Let $\{G_\alpha\}_{\alpha \in I}$ be an open cover



Since B is closed, B^c is open.

So $A = (A \cap B) \cup (A \setminus B)$

$\{B^c\}$ is an open cover for $A \setminus B$ and $\{G_\alpha\}_{\alpha \in I}$ is an open cover for $A \cap B$

Therefore $A \subseteq (\bigcup_{\alpha \in I} G_\alpha) \cup B^c$

Since A is compact and so there is a finite subcover say $\{G_{\alpha_1}, G_{\alpha_2} - - - - G_{\alpha_n}, B^c\}$ for A

Since $B^c \cap (A \cap B) = \emptyset$, $\{G_{\alpha_1}, G_{\alpha_2} - - - - G_{\alpha_n}\}$ is a cover for $A \cap B = E$

Thus E is compact.

Compactness in a Hausdroff Space

We ist prove a very useful lemma.

Lemma# Let (X, T) be a topological space.

(i) If B is a compact subset in (X, T) and x be a point in X, such that x can be strongly separated from every point y in B, then x and B can also be strongly separated in $(X\ T)$

(ii) If A and B are two compact subsets in $(X\ T)$ such that every point x in A can be strongly separated from every point y in B, A & B can be strongly separated.

Proof# (i) Let $U_y(x)$ & $V_x(y)$ seperate strongly the point x from a point $y \in B$. Then collection $\{V_x(y) : y \in B\}$ form an open covering of B.

Since B is compact, this covering will contain a

$V_x(y_1), V_x(y_2) \ldots V_x(y_n)$. Let $U_{y_1}(x), U_{y_2}(x) \ldots U_{y_n}(x)$ be the corresponding open sets containing x. Let

$$U_B(x) = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_n}(x)$$

and

$$V_x(B) = V_{y_1}(x) \cup V_{y_2}(x) \cup \ldots \cup V_x(y_n)$$

Then $x \in U_B(x)$ and $B \subseteq V_x(B)$

$\because U_{y_i}(x) \cap V_x(y_i) = \phi \quad \forall i = 1, 2 \ldots n$

and $U_B(x) \subseteq U_{y_i}(x)$, for $i = 1, 2 \ldots n$

It follows that

$$\bigcup_B U_B(x) \cap V_x(B) = \phi$$

Thus x & B are separated strongly by the pair of disjoint open sets $U_B(x)$ & $V_y(B)$ in (X, τ)

(ii) In part (i) the collection $\{U_B(x_i) : x \in A\}$ form a covering of A. Since A is compact, this covering has a finite sub-covering $\{U_B(x_i) : i = 1, 2 \ldots m\}$ let $V_{x_1}(B), V_{x_2}(B) \ldots V_{x_m}(B)$ be the corresponding open sets containing B. Then

$$U(A) = U_B(x_1) \cup \ldots \cup U_B(x_m) \text{ and.}$$

$$V(B) = V_{x_1}(B) \cap \ldots \cap V_m(B)$$

are two disjoint open sets because.

$$U_B(x_i) \cap V_{x_i}(B) = \phi \quad \forall i = 1, 2 \ldots m$$

and $V(B) \subseteq V_{x_i}(B) \quad \forall i = 1, 2 \ldots m.$

Thus $U(A)$ & $V(B)$ separate A and B strongly

Theorem# In a Hausdorff space, any point and disjoint compact subspace can be separated by open sets, in the sense that they have disjoint neighbourhoods

Proof# Let X be a Hausdorff space, x a point in X and C a compact subspace of X which does not contain x.

We construct a disjoint pair of open sets G and H such that

$$x \in G \quad \text{and} \quad C \subseteq H.$$

Let y be a point in C. Then $x \neq y$ because $\{x\}$ and C are disjoint sets.

$\because$ X is Hausdorff space

$\therefore$ x and y have disjoint neighbourhoods G_x & H_y

Now Collection $\{H_y : y \in C\}$ forms a cover for C; and since C is compact, some finite subclass $\{H_1, H_2, \ldots H_n\}$ is also a cover for C i.e.

$$C \subseteq \bigcup_{i=1}^{n} H_i$$

If $G_1, G_2 \ldots G_n$ are corresponding neighbourhoods of x, corresponding to the H_i's, we put

$$G = \bigcap_{i=1}^{n} G_i$$

and $$H = \bigcup_{i=1}^{n} H_i$$

Then $C \subseteq H$ $\quad x \in G$ and $G \cap H = \phi$

Thus the point x and the compact set C can be separated in the sense that they have disjoint nghds.

Theorem# Set X be a Hausdorff space, C a compact subset of X and x an element of X which is not in C. Then there are disjoint open sets U_x and V_x such that

$$x \in U_x \quad \& \quad C \subseteq V_x$$

Proof# Suppose that C is a compact subset of X and $x \in X \setminus C$. Then

For each $y \in C$, $y \neq x$

$\because$ X is a Hausdorff space

$\therefore$ There open sets U_{xy} and V_y such that

$$x \in U_{xy}, \quad y \in V_y \quad \text{and} \quad U_{xy} \cap V_y \neq \emptyset$$

Here for each $y \in C$ and for a fixed $x \in X \setminus C$, we have corresponding set V_y, s.

Now $\{V_y \cap C : y \in C\}$

is an open cover for C because.

$$C \subseteq \bigcup_{y \in C} V_y$$

$$\Rightarrow C \cap C \subseteq \left(\bigcup_{y \in C} V_y\right) \cap C$$

$$C \subseteq \bigcup_{y \in C} (V_y \cap C)$$

$\because$ C is compact

$\therefore$ This open cover has a finite subcover

$V_{y_1} \cap C, V_{y_2} \cap C, \ldots V_{y_n} \cap C$

Let the corresponding sets containing x be

$$U_{xy_1}, U_{xy_2} \ldots \ldots U_{xy_n}$$

Let

$$U_x = \bigcap_{i=1}^{n} U_{xy_i}$$

and $V_x = \bigcup_{i=1}^{n} V_{y_i}$

Then $x \in U_x$, $C \subseteq V_x$

and

$$U_x \cap V_x = U_x \cap \left(\bigcup_{i=1}^{n} V_{y_i}\right)$$

$$= \bigcup_{i=1}^{n} \left(U_x \cap V_{y_i}\right)$$

$$= \phi$$

as required

\# Any arbitrary subset of a compact space may not be compact. #

Theorem# Every compact subset of Hausdorff space is closed

Proof# Let X be compact and C a compact subset of X. We show that C' is open.

Let $x \in C'$ $\quad x \in C'$

$\because C' \cap C = \phi$

$\therefore$ Any $y \in C$, is distinct from x i.e $x \neq y$ $\forall y \in C$.

$\because X$ is a Hausdorff space

$\therefore$ For any $x \in C$, there are disjoint open sets U_{xy} and V_y containing x & y such that

$$U_{xy} \cap V_y = \phi$$

Now $\quad x \in V_y \quad \forall y \in C$

$$\Rightarrow C \subseteq \bigcup_{y \in C} V_y$$

$$\Rightarrow C \cap C \subseteq \left(\bigcup_{y \in C} V_y\right) \cap C$$

$$\Rightarrow C \subseteq \bigcup_{y \in C} (V_y \cap C)$$

$\Rightarrow$ Collection $\{V_y \cap C : y \in C\}$ is an open cover of C. Since C is compact, this open cover

will have a finite subcover $\{V_{y_i} \cap C : i = 1, 2 \ldots n\}$, where $y_1, y_2 \ldots y_n$ are in C.

Thus

$$C \subseteq \bigcup_{i=1}^{n} (V_{y_i} \cap C) \qquad \text{equality}$$

Now corresponding to each y_i, let U_{xy_i} be the open set containing x. Then

$$x \in U_x = \bigcap_{i=1}^{n} U_{xy_i}$$

and

$$U_x \cap C = U_x \cap \left(\bigcup_{i=1}^{n} V_{y_i} \cap C\right)$$

$$\subseteq U_x \cap \left(\bigcup_{i=1}^{n} V_{y_i}\right)$$

$$\subseteq \bigcup_{i=1}^{n} (U_x \cap V_{y_i}) = \phi$$

$$\Rightarrow U_x \cap C = \phi \qquad (\because U_x \cap V_{y_i} = \phi$$

$$\Rightarrow U_x \subseteq C'$$

$$\Rightarrow x \in U_x \subseteq C'$$

Now taking union over all element of C', we have.

$$\bigcup_{x \in C'} \{x\} \subseteq \bigcup_{x \in C'} U_x \subseteq C'$$

$$C' \subseteq \bigcup_{x \in C'} U_x \subseteq C'$$

$$C' = \bigcup_{x \in C'}$$

$\Rightarrow C'$ is the union of open sets and hence is open, so C is closed

OR

$$x \in U_x \subseteq C'$$

$\Rightarrow$ Any arbitrary point x of C' is an interior point of C' and hence C' is open, C is closed

Corollary # (i) Every compact set of real numbers is closed and bounded

(ii) Every compact subset of R^n is closed and bounded

Proof # (i) Since R is Hausdorff, a compact subset C of R must be closed.

Moreover collection of intervals

$\{I_n = (-n, n) : n \in N\}$ form an open covering of C and so a finite number of these must cover C. Hence C must be bounded.

(ii) Let C be a compact subset of R^n. Since R^n is Hausdorff, C is closed.

Consider now the open balls

$U_k = B(\underline{0}, k)$

where $\underline{0} = (0, 0, \ldots 0)$ and k is a natural number. Then $\{U_k : k \in N\}$ is an open cover for R^n. Since C is compact, there are finite numbers of open balls $U_{n_1}, U_{n_2} \ldots U_{n_p}$ such that

$$C \subseteq \bigcup_{i=1}^{p} U_{n_i}$$

Let $n = \max(n_1, n_2 \ldots n_p)$

Then $C \subseteq B(0\ n)$

Hence C is bounded.

Theorem # In a Compact Hausdorff space, the compact subsets are same as the closed subsets.

Proof # ∵ Any closed subspace of a compact space is compact

∴ Every closed subset of a compact Hausdorff space is compact set

Also since every compact subset of a Hausdorff space is compact, all compact sets in Hausdorff space are compacts. Thus compact subsets are same as the closed subsets.

Theorem # A Compact Hausdorff space is normal and hence T_4- space.

Proof # Let X be a compact Hausdorff space and C_1, C_2 be any two disjoint closed sets of X.

∵ Every closed subset of a compact space is compact

∴ C_1, C_2 are Compact

Since $C_1 \cap C_2 = \phi$, therefore any pair points x, y with $x \in G_1$, $y \in G_2$, $x \neq y$

∵ X is Hausdorff

∴ By T_2- there exists two disjoint open sets ~~for each pair~~ $G_1(x,y)$ and $H(x,y)$ containg x and y respectively.

For each fixed point $x \in C_1$, the collection

$\{H(x,y) : y \in C_2\}$ forms an open covering for C_2.

Since C_2 is compact, there must be a finite subcovering

$\{H(x,y_i) : i = 1,2,3 \ldots n\}$

let the corresponding open sets containg x and disjoint from $H(x,y_i)$ be $G_1(x,y_i) : i = 1,2 \ldots n$ and

$$U_x = \bigcap_{i=1}^{n} G_1(x,y_i) \quad \text{and} \quad V_x = \bigcup_{i=1}^{n} H(x,y_i)$$

Then $x \in U_x$ & $C_2 \subseteq V_x$

and $U_x \cap V_x = \phi$

Because

$$U_x \cap V_x = U_x \cap \left(\bigcup_{i=1}^{n} H(x,y_i)\right)$$

$$= \bigcup_{i=1}^{n} [U_x \cap H(x,y_i)]$$

$$= \phi$$

∵ $U_x \cap H(x,y_i)$

$\left(\bigcap_{i=1}^{n} G_1(x,y_i)\right) \cap H(x,y_i) = \phi \ \forall i$

So for each $x \in C_1$, we open sets U_x & V_x such that $x \in U_x$, $C_2 \subseteq V_x$

Now the collection $\{U_x : x \in C_1\}$ forms an open covering for the compact set C_1. There must be a finite subcovering $\{U_{x_i} : i = 1, 2, \ldots, m\}$ such that

$$C_1 \subseteq \bigcup_{i=1}^{m} U_{x_i}$$

Let the corresponding open sets supersets of C_2 be

$V_{x_i} : i = 1, 2 \ldots m$ and.

$$G = \bigcup_{i=1}^{m} U_{x_i} \qquad H = \bigcap_{i=1}^{m} V_{x_i}$$

Then $C_1 \subseteq G$, $C_2 \subseteq H$ and.

$$G \cap H = \left(\bigcup_{i=1}^{m} U_{x_i}\right) \cap H$$

$$= \bigcup_{i=1}^{m} (U_{x_i} \cap H)$$

$$= \phi$$

Thus. X is normal space. Also since X is Hausdorff, it is T_1- space and hence X is - T_4-space.

Theorem # Any compact Hausdorff space is regular

Proof # Let X be a compact. Hausdorff space; C be a closed subset of X and x be any point of X outside C.

$\because$ Every closed subset of a (Hausdorff) compact space is compact

$\therefore$ C is compact.

Now $x \neq y$ $\forall y \in C$

$\because$ X is Hausdorff space.

$\therefore$ for any point y in C, there are open sets G_{xy}, H_y containing x & y respectively and.

$$G_{xy} \cap H_y = \phi$$

Now the collection $\{H_y : y \in C\}$ is an open covering for C

$\because$ C is compact

$\therefore$ there is a finite subcovering $\{H_{y_i} : i = 1, 2 \dots n\}$.

Let the coressponding open sets containg x be.

$G_{xy_i} : i = 1, 2 \dots n$ and.

$G_x = \bigcap_{i=1}^{n} G_{xy_i}$ $\qquad H = \bigcup_{i=1}^{n} H_{y_i}$

Then $x \in G_x$ $\qquad C \subseteq H$ and $G_x \cap H = \phi$

Hence X is regular

Theorem # Any Continuous bijective function from a compact space X to a Hausdorff space Y is also open and necessarily a Homeomorphism.

Proof # Let X be compact and Y a Hausdorff space. Let $f : X \longrightarrow Y$ be a continuous bijective function. To prove that f is homeomorphism we are to only show that $\bar{f}^1$ is continuous i.e $(f^{-1})^{-1}(G) = f(G)$ is in Y for every open set G in X or $(\bar{f}^1)^{-1}(F) = f(F)$ is closed in Y for every closed set F in X.

Let F be any closed set in X which is Compact.

$\because$ Any closed subset of a compact space is compact

$\therefore$ F is compact

$\because$ Continuous image of a Compact set is Compact

$\therefore$ $f(F)$ is a compact subset of Y, which is Hausdorff

$\because$ Every Compact subset of a Hausdorff space is closed

$\therefore$ $f(F)$ is closed in Y

Hence $\bar{f}^1$ is continuous - Thus f is homeomorphism.

OR

We can prove that f is homeomorphism by showing that f is open..

Let G be any open set in X. Then $G^c = X \setminus G$ is closed

subset of X, which is Compact.
∵ Closed subset of a Compact space subset of is Compact
∴ $f(G^c)$ is Compact subset of Y, which.
is Hausdorff
Hence $f(G^c)$ is closed.
⇒ $Y \setminus f(G^c)$ is open.
∵ f is one-one and onto
∴ $Y \setminus f(G^c)$

$$= f(X) - f(X-G) = f(X) - [f(X) - f(G)]$$
$$= f(X) - [f(G)]^c$$
$$= [[f(G)]^c]^c = f(G)$$

⇒ f(G) is open
⇒ f is homeomorphism.

Locally Compact Space

A top-space (X, T) is said to be locally compact iff each point of X is contained in a compact neighbourhood.

A slightly different definition of local Compactness is also in use today

A top-space is said to be locally Compact if each point of it is contained in a neighbourhood whose closure is compact

Examples

(1) Since each Compact space is a Compact neighbourhood of each of its points, every Compact space is Locally Compact.

(2) R^n is locally Compact, because any open sphere centred on any point is a nbd of the point whose closure being a closed and bounded subspace of R^n, is compact

3)) Any discrete space is locally compact because each singleton is compact and each point space is contained in singleton neighbourhood which are compact.

Theorem # A closed subset of a locally compact space is locally compact

Proof # Let (X, T) be a locally compact space and A be any closed subset of X

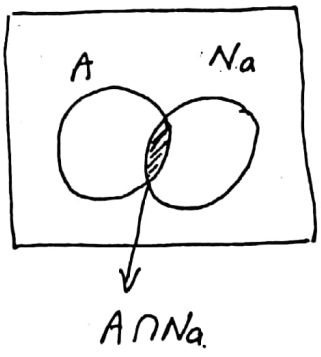
A will be compact if each point of A is contained in a compact neighbourhood in relative space (A, T_A).

Let a be any element of A. Then a, being an element of X is contained in some compact neighbourhood N_a in X. But then $A \cap N_a$ is a neighbourhood of A in relative space (A, T_A)

Now let $E = A \cap N_a$

$\because$ A is closed

$\therefore$ A^c is an open set in X



$N_a = (A \cap N_a) \cup (N_a \setminus A)$

$\{A^c\}$ is an open cover for $N_a \setminus A$

Let $\{G_\alpha\}_{\alpha \in I}$ be an open cover for E

Then $N_a \subseteq \left(\underset{\alpha \in I}{\cup} G_\alpha\right) \cup A^c$

$\Rightarrow \{G_\alpha : \alpha \in I, A^c\}$ is an open cover for N_a in X.

$\because$ N_a is compact

$\therefore$ There is a finite subcover $\{G_{\alpha_1}, G_{\alpha_2} \ldots G_{\alpha_n}, A^c\}$ for N_a

$\therefore A^c \cap E = \phi$

$\therefore \{G_{a_1}, G_{a_2} \cdots G_{a_n}\}$ is a finite subcover for $E = A \cap N_a$.

$\Rightarrow A \cap N_a$ is a compact neighbourhood of a.

$\because a$ is an arbitrary (neighbour) element of A

$\therefore$ each element of A is contained in a compact neighbourhood.

Hence A is locally compact

Locally Compact Hausdorff Spaces

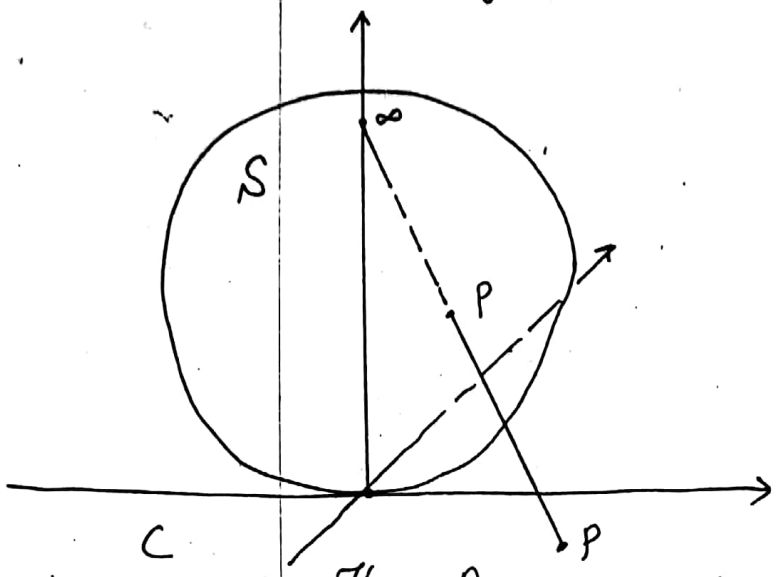
we have defined locally compact spaces. These often arise in the applications of topology to Geometry and analysis. Since the spaces which arise in the applications of topology to geometry and analysis are almost always Hausdorff space, we restrict ourself to locally compact Hausdorff spaces.

The main fact about locally compact Hausdorff space is that it can be converted into a compact Hausdorff space by suitably adjoining a single point. e.g.

The complex C is enlarged by adjoining to it an "ideal point" called the point at infinity and is denoted by ∞. This ideal point can be considered as any object not in C and the larger set $C \cup \{\infty\}$ is denoted by C_∞. C_∞ is called extended complex plane when the neighbourhoods of ∞ (other than C_∞ itself) are taken to be the complements in C_∞ of the closed and bounded subsets (i.e the compact spaces) of C

The figure below gives an easy way of visualizing the extended complex plane. In figure, the surface S of sphere of radius $\frac{1}{2}$ is rested

tangentially on C at the origin.



The Riemann sphere.

It is customary to call the point of contact the south pole and the opposite point the pole. The indicated projection from the north pole sets up a homomorphism between S minus its north pole and C, so from the topological point of view, S minus its north pole can be regarded essentially identical with the complex plane C. The north pole can be considered to be the point at infinity and passing from C to C_∞ amounts to using the point ∞ to plug up the hole in C at the north pole. When S identified in this manner with the extended complex plane, it is usually called the Riemann sphere. In brief, the locally compact Hausdorff space C has been made into the compact Hausdorff space S by adding the single point ∞.

One Point Compactification of a Top. Space

Let X be an arbitrary locally compact Hausdorff space. Let ∞ be an object not in X and $X_\infty = X \cup \{\infty\}$. We define a topology on X_∞ by specifying the following as open sets

open subsets of X regarded as subsets of X_∞

(ii) The complements in X_∞ of compact subspaces of X OR all subsets G_i of X_∞ such that $X_\infty \setminus G_i$ is closed & compact subset of X

(iii) the full space X_∞

If we keep in mind that compact subspace of a Hausdorff space is closed, then it is easy to show that the class of sets is actually a topology on X_∞ and also the given topology on X equals its relative topology as a subspace of X_∞.

The following are the main properties of the topological space X_∞

(I) X_∞ is Compact #

Let $\{G_i\}_{i\in I}$ be an open cover of X_∞. If X_∞ occurs among the G_i's, then $\{G_i\}$ clearly has a finite subcover namely, $\{X_\infty\}$. We assume that $\{G_i\}_{i\in I}$ contain sets of type (i) or (ii).

At least one G_i, say G_{i_0} must contain the point ∞, and this set is necessarily of type (ii). Its complement G_{i_0}' is thus a compact subspace of X. Clearly $\{G_i \cap X\}_{i\in I}$ will be an open cover. So it will have a finite sub-cover say $\{G_1 \cap X, G_2 \cap X, \ldots\ldots G_n \cap X\}$ i.e

$$G_{i_0}' \subseteq \bigcup_{j=1}^{n} \{G_j \cap X\} \subseteq \bigcup_{j=1}^{n} G_j \qquad G_{i_0} \cup G_{i_0}' = X_\infty$$

$$\therefore G_{i_0} \cup G_{i_0}' = X_\infty$$

Therefore

$$G_{i_0} \cup G_{i_0}' \subseteq \left(\bigcup_{j=1}^{n} G_j\right) \cup G_{i_0}$$

$$\Rightarrow \quad X_\infty \subseteq \left(\bigcup_{j=1}^{n} G_j\right) \cup G_{i_0}$$

Thus $\{G_{i_0}, G_1, G_2 \ldots G_n\}$ is a finite sub-cover of X_∞. Thus X_∞ is Compact.

(2) **X_∞ is Hausdorff #**

Since X is Hausdorff, any pair of distinct points in X_∞ both of which lie in X can be separated by open sets of X and thus can be separated by open subsets of X_∞ of type (1). It is therefore sufficient to show that any point x in X and the point ∞ can be separated by open sets in X_∞. Since X is locally compact, x has a nbd G whose closure $\bar{G}$ in X is compact. Now G and $\bar{G}'$ are open sets in X_∞ and

$G \cap \bar{G}' = \phi$.

Also $x \in G$ and $\infty \in \bar{G}'$

So X_∞ is Hausdorff.

The Compact Hausdorff space X_∞ associated with locally Compact space X in the manner above is called one-point compactification of X and the point ∞ is called the point at infinity.

Comments (1) It is easy to see that the locally compact Hausdorff space X is compact $\Leftrightarrow$ ∞ is an isolated point point of X_∞.

(2) The one-point Compactification is useful mainly in simplifying the proofs of Theorems about locally Compact Hausdorff spaces. As an example, any space X of this kind is easily see to be completely regular; for X is a subspace of X_∞, which is compact and therefore completely regular and every subspace of a completely regular space is completely regular. Accordingly, if x is a point of X, and G a neighbourhood of x which does not equal the full space, then there exists a continuous real function $f: X \to [0,1]$ such that

$$f(x) = 1 \qquad f(G') = 0.$$

This fact can be generalised, again by using the

the one point compatification, to the case in x is replaced by an arbitrary compact subspace of X

Theorem # Let X be a locally compact Hausdorff space, Let C be a compact subspace of X, and let G be a nbd of C which does not equal fullspace. Then there exist a continuous function $f: X \to [0,1]$ such that

$$f(C) = 1 \quad \& \quad f(G') = 0$$

Proof # Let X_∞ be the one point compactification of X. Then C and G' are disjoint and closed subspaces of X_∞ and by Urysohn's lemma there exists a continuous real function $g: X_\infty \to [0,1]$ such that

$$g(C) = 1 \qquad g(G') = 0$$

If f is restricted of g to X, then f has the required properties.

Locally Compact Space

A top- space (X, T) is compact if for each $x \in X$, there is an open set G containg x such that $\bar{G}$ is compact.

Theorem # A top- space X is locally compact iff the collection of open sets with compact closures form a base for topology of X

Proof # Let X be locally compact and $\{G_\alpha, \alpha \in I\}$ be collection of open sets whose closures are compact.

$\because$ X is compact

$\therefore$ each element x in X is contained in some open set whose closure is compact i.e for each $x \in X$

$\exists$ G_α for some $\alpha \in I$ such that

$x \in G_\alpha$ a

Let G be any open set. Then for any $y \in G$ $\exists$ G_{α_0} such that

$$y \in G_{\alpha_0}$$

Now we prove that $G_{\alpha_0} \subseteq G$.

Let G

Conversely let the collection of all opens with compact closures be a base for the topology. Then clearly every point of X is contained in an open set whose closure is compact. Hence X is locally compact

Examples #

(i) Every discrete space is locally compact

(ii) The real line R is locally compact but not compact.

(iii) The homomorphic image of a compact space is locally compact.

Here let $f: X \longrightarrow Y$ be a homomorphism from a compact space X to a space Y. Then f is continuous and so $f(X)$ is compact. Since every compact space is locally compact, $f(X)$ is locally compact.

Theorem # Let X be a locally compact space and C, a compact subset of it. Prove that there is an open set G such that

$$C \subseteq G \quad \text{and} \quad \bar{G} \text{ is compact}$$

Proof # $\because$ C is compact and so locally compact

$\therefore$ for each $x \in C$, there is an open set O_x with $\bar{O}_x$ compact, $x \in O_x$

Then collection $\{O_x : x \in C\}$ is an open cover for C. Since C is compact, this open cover has a

finite subcover $\{O_{xi} : i = 1, 2 \ldots n\}$

Let $O = \bigcup_{i=1}^{n} O_{xi}$

Then

$C \subseteq O$

Also $\bar{O} = \bigcup_{i=1}^{n} \bar{O}_{xi}$, being the uni of finite no of compact sets is compact.

Theorem # Let $f: X \to Y$ be an injective function, continuous from a compact space X to a Hausdorff space Y. Then X, $f(X)$ are homomorphic

Proof # Since $f: X \to y$ is continuous, $f: X \to f(X)$ is also continuous

Also f is injective as given and obviously surjective, so $f: X \to f(X)$ is bijection

Now we prove that $f: f(X) \to X$ is continuous.

Let F be a closed subset of X

Then F being closed subset of a compact space is compact.

$\Rightarrow f(F)$ is compact because continuous image of a compact subset is compact

$\Rightarrow f(F)$ is closed because compact subset of a Hausdorff space is closed

$\Rightarrow (f^{-1})^{-1}(F) = f(F)$ is closed

$\Rightarrow f^{-1}: f(X) \to X$ is continuous

Hence X, $f(X)$ are homomorphic.

Theorem # Product of two Compact spaces is compact

Proof # Let X, Y be two compact spaces, we prove that $X \times Y$ is compact.

Let $\{ W_\gamma : \gamma \in I_W \}$ be an open cover for $X \times Y$

$\Rightarrow W_\gamma = U_\alpha \times V_\beta$ where U_α is open in X & V_β open in Y. Consider the collections

$\{ U_\alpha : \alpha \in I_U \}$ & $\{ V_\beta : \beta \in I_V \}$. We prove that these collections are open cover for X & Y respectively as under

Let $x \in X$, $y \in Y$, $(x, y) \in X \times Y$.

Then $(x, y) \in U_\alpha \times V_\beta$ ... for some $\alpha \in I_U$, $\beta \in I_V$.

because $\{ U_\alpha \times V_\beta : \alpha \in I_U, \beta \in I_V \} = \{ W_\gamma : \gamma \in I_W \}$ is an open cover for $X \times Y$

$\Rightarrow x \in U_\alpha$ for some $\alpha \in I_U$ & $y \in V_\beta$ for some $\beta \in I_V$

$\Rightarrow x \in \bigcup_{\alpha \in I_U} U_\alpha$ & $y \in \bigcup_{\beta \in I_V} V_\beta$

$\Rightarrow X = \bigcup_{\alpha \in I_U} U_\alpha$ & $Y = \bigcup_{\beta \in I_V} V_\beta$

$\Rightarrow \{ U_\alpha / \alpha \in I_U \}$ & $\{ V_\beta : \beta \in I_V \}$ are open covers of X & Y respectively. But X & Y are both compact. Therefore $\exists$ finite open covers $\{ U_{\alpha_1}, U_{\alpha_2} \ldots U_{\alpha_k} \}$ and $\{ V_{\beta_1}, V_{\beta_2}, \ldots V_{\beta_m} \}$. where $\alpha_i \in I_U$ $\forall 1 \le i \le k$ and $\beta_j \in I_V$ $\forall 1 \le j \le m$

Now we prove that the collection

$\{ U_{\alpha_i} \times V_{\beta_j} : 1 \le i \le k, \ 1 \le j \le m \}$ is a finite subcover of $\{ W_\gamma : \gamma \in I_W \}$ for $X \times Y$

Which is proved as under

Consider any point $(x, y) \in X \times Y$.

$\Rightarrow x \in X$ $\quad y \in Y$

$\Rightarrow (x, y) \in U_{\alpha_i} \times V_{\beta_j}$ for some $\alpha_i \in I_U$, $\beta_j \in I_V$

$\Rightarrow X \times Y \subseteq \bigcup_{\substack{1 \le i \le k \\ 1 \le j \le m}} (U_{\alpha_i} \times V_{\beta_j})$

Which proves that $X \times Y$ is compact

Cluster Point OR Limit Point

A point p in a top-space (X, T) is called a cluster point of an infiinte sequence $\{x_n\}$ if for every open set U containg p $\exists$ a +ve integer n_0 such that

$$x_n \in U \quad \forall n \geq n_0$$

Bolzano-Weirstrass Property

A top-space X is said to have the B.W property if every infinite subset of X has limit point in X

Countably Compact Top-Space

A top-space X is said to be Countably compact iff every countable open cover has a finite subcover.

Comments # Since every space satisfying the second axiom of countability has the property that every open covering has a countable subcover, it follows that countable compactness is equivalent to compactness in the presence of second axiom of countability.

Theorem # In a 2nd axiom space, every open covering of a subset is reducible to a countable i.e every 2nd countable space is a Lindelöf space.

Proof # Suppose $\{B_n\}$ be a countable base for 2nd axiom space and E be a subset of X.

Let C be an open covering for E. $N(C)$ be the countable collection of integers n such that $B_n \subseteq G$ for some $G \in C$. Then

$$B_n \subseteq G_n, \quad \forall n \in N(C) \text{ \& } G_n \in C$$

· The family $\{G_n : n \in N(C)\}$ is clearly countable subcollection of C. We prove that this is an open covering of E.

Let $x \in E$.

Since C is an open covering of E, $x \in G$ for some $G \in C$. From the definition of base, we have $x \in B_n \subseteq G$ for some integer n

$\because$ $N(C)$ is collection of all such integers for which B_n is contained in some $G \in C$

$\therefore$ $n \in N(C)$

$\Rightarrow$ $x \in B_n \subseteq G_n$

$\Rightarrow$ $E \subseteq \cup\{G_n : n \in N(C)\}$

Thus proved

Theorem # Every Compact space is countably compact

Proof # Since in a Compact space every open. Cover has a finite subcover, therefore every countable cover will also have a finite subcover

Theorem # A top space (X, T) is countably Compact iff every countable collection of closed set possessing F.I.P, itself has a non-empty intersection

Proof # Let $(X\ T)$ be compact and $\{F_i : i \in I\}$, where I is a countable set, be a countable class of closed sets with F.I.P. We are to prove that

$$\bigcap_{i \in I} F_i \neq \emptyset$$

Let on the contrary $\bigcap_{i \in I} F_i = \emptyset$

Complementing on both side

$$\left(\bigcap_{i\in I} F_i\right)' = X$$

$$\bigcup_{i\in I} F_i' = X \quad \text{(De. Morgan Laws)}$$

where each F_i' is open.

Thus $\{F_i' : i\in I\}$ is an countable open cover for X.

$\because$ X is countably compact

$\therefore$ This open cover has a finite subcover

$\{F_j' : j = 1, 2 \cdots n\}$ i.e

$$X = \bigcup_{j=1}^{n} F_j'$$

Complementing on both sides

$$\phi = \left(\bigcup_{j=1}^{n} F_j'\right)^c$$

$$= \bigcap_{j=1}^{n} F_j$$

$\Rightarrow$ finite Sub collection $\{F_j : j = 1, 2 \cdots n\}$ of $\{F_i : i\in I\}$ has an empty intersection. But this is a Contradiction to F.I.P. Hence

$$\bigcap_{i\in I} F_i \neq \phi$$

Conversely suppose that every countable class of class of closed sets with F.I.P has non-empty intersection

Let $\{G_\alpha : \alpha \in I\}$, where I is a countable set, be a countable cover for X i.e

$$\bigcup_{\alpha\in I} G_\alpha = X$$

$\therefore$ Complementing

$$\bigcap_{\alpha\in I} G_\alpha' = \phi$$

$\Rightarrow \{G_\alpha' : \alpha \in I\}$ forms a countable collection of closed sets with F.I. empty intersection. It follows from the given condition that the family $\{G_\alpha' : \alpha \in I\}$ can not possess F.I.P. Hence a certain finite subcollection say $\{G_i' : i = 1, 2 \ldots n\}$ must have an empty intersection i.e

$$\bigcap_{i=1}^{n} G_i' = \phi$$

Complementing

$$\bigcup_{i=1}^{n} G_i = X$$

$\Rightarrow \{G_i : i = 1, 2, \ldots n\}$ forms a finite subcover for X. Hence (X, τ) is countably compact.

Theorem # A top-space X is countably compact iff every descending chain of non-empty closed sets

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ has a non-empty intersection in (X, τ)

Proof # Let (X, τ) be countably compact and $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ be a descending chain of non-empty closed sets in (X, τ). Then $\{F_i : i = 1, 2 \ldots\}$ is a countable class of closed sets (X, τ), possessing F.I.P. Hence $\bigcap F_i$ is non-empty by above theorem.

Conversely let the condition hold in (X, τ). Let $\{k_i : i = 1, 2 \ldots\}$ be a countable aggregate of closed sets having F.I.P in (X, τ). Let

$F_i = k_1 \cap k_2 \cap \ldots \cap k_i$ for $i = 1, 2 \ldots$, Then

$F_1 = k_1$

$F_2 = k_1 \cap k_2 \subseteq F_1 = k_1$

$F_1 \supseteq F_2$

$F_3 = k_1 \cap k_2 \cap k_3 = F_2 \cap k_3 \subseteq F_2$

Hence

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a descending chain of non-empty closed sets in (X, T). Hence. by Hypothesis $\cap\{F_i : i = 1, 2, \ldots\}$ is non-empty. Consequently $\cap\{k_i : i = 1, 2 \ldots\} = \cap\{F_i : i = 1, 2 \ldots\}$ is non empty. Hence the space (X, T) is Countably Compact.

Theorem # A top-space (X, T) is countably compact iff every infinite sequence in X has a cluster point in X

Proof # Let $\{x_i : i = 1, 2 \ldots\}$ is an infinite sequence in (X, T) with no cluster point. Then for each point $p \in X$, $\exists$ an open set U_p such that

$$p \in U_p \text{ and } U_p \cap \{x_{r+1}, x_{r+2}, \ldots\} = \emptyset$$

For each +ve integer n, let G_n be the union of all open sets U_p which do not contain any term of $\{x_{n+1}, x_{n+2}, \ldots\}$. Then $\{G_1, G_2 \ldots\}$ forms an countable open covering of X, for which there do not exist any finite subcovering. Hence the space (X, T) can not be countably compact.

Conversely let every infinite sequence in (X, T) has a cluster point in (X, T) and (X, T) is not Countably Compact. Then there is a Countable open cover $\{G_1, G_2 \ldots\}$, for which there does not exist any sub-covering Let $H_1 = G_1$ and $x_1 \in H_1$. For $n > 1$, let H_n be the ist one of $G_2, G_3 \ldots$ which is not contained in

$H_1 \cup H_2 \cup \cdots H_{n-1}$ and

let $x_n \in H_n - (H_1 \cup \cdots \cup H_{n-1})$

Let $p \in X$, then there exists a positive integer r, such that $p \in H_r$. Then p is not a cluster point

of the infinite sequence $\{x_i : i = 1, 2, \ldots\}$, since

$$H_r \cap \{x_{r+1}, x_{r+2}, \ldots\} = \phi.$$

Thus the infinite sequence $\{x_i : i = 1, 2 \ldots\}$ has no cluster point in X

Hence (X, T) is Countably Compact

Theorem# A top-space (X, T) is Countably compact iff every infinite subset in X has an ω-accumulation point in X

Proof# Let X be a Countably compact space. Let X Contains an infinite subset S, which has no ω-accumulation point in X. Then there exists a denumerable subset T of S such that T has no ω-accumulation point in X. Hence for every $p \in X$, there exists an open set G_p such that

$$p \in G_p \text{ and } G_p \cap T \text{ is a finite set.}$$

For every finite subset F of T (including the case $F = \phi$), let $G_F = \cup\{G_p : G_p \cap T = F\}$.

The aggregate of sets $\{G_F\}$ forms a countable open covering of X. But since each G_F intersects the infinite set T in only finitely many points, for the countable open covering $\{G_F\}$ of X, there does not exist any finite subcovering. Hence space (X, T) is not Countably compact which is a contradiction. Thus every infinite subset has a ω-accumulation point.

Conversely suppose that every infinite subset of X has an ω-accumulation point. Let (X, T) be not Countably compact. There there exists a countable open covering $\{V_i : i = 1, 2, \ldots\}$ of X, for which there is no finite subcovering. Then for any +ve integer n, $X - \{V_1 \cup V_2 \cup \ldots \cup V_n\}$ is non-empty. Let $x_n \in X \setminus (V_1 \cup V_2 \cup \ldots \cup V_n)$, for $n = 1, 2 \ldots$

Let p be any point in X, then $p \in V_m$, for some m. The point p can not be an ω-accumulation point of the infinite subset $S = \{x_1, x_2 \ldots\} \subseteq X$, since $p \in V_m$ and $V_m \cap S$ is a finite subset. Hence (X, T) must be Countably compact.

ω-Accummulation Point

A point p of a top-space (X, T) is called an ω-accumulation point or an ω-limiting point of a subset A of X if every nbd (open set) containing p intersects A in infinitely many points.

Further if every nbd of p intersects A in a non-countable infinite set of points, then p is called a point of condensation of the set A. Presumably, A must be a non-Corentable set in this case.

Theorem # (a) # A subspace of a countably compact need no be countably compact.

(b) # Every closed subspace of a countably compact space is countably compact.

(c) # The union of a finite collection of countably compact subspaces of a topological space is a countably compact subspace.

Proof # (a) We know that $[0\ 1]$ is compact by Heine-Borel theorem and hence is countably compact. The subspace $(0, 1)$ of $[0\ 1]$ is, however, not countably compact

(b) # Let (X, T) be countably compact space and A be a closed subset in $(X\ T)$. Let $\{F_\alpha : \alpha \in I\}$, I is countable, be a countable collection of closed sets in (A, T_A), possessing F.I.P. Then each F_α is equal to intersection of A with some closed set in X and is therefore closed in X. Thus $\{F_\alpha : \alpha \in I\}$ is a countable family of closed sets in X

in (X, T), possessing finite intersection property. Since X is Countably compact, Therefore

$$\bigcap_{\alpha \in I} F_\alpha \neq \phi$$

Hence (A, T_A) is Countably compact

(C) Let $Y_1, Y_2 \cdots Y_n$ be a finite collection of Countably compact spaces of topological space (X, T).

Let $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cdots \cup Y_n$

Let V be a Countable open covering for Y, then V is also a Countable open covering for each Y_i. Since each Y_i is Countably Compact, there exists a finite sub-cover

$U_i = \{U_{i,1}, U_{i,2} \cdots U_{i,r_i}\}$ of V for each Y_i

Then the sets belonging to $U_1, U_2 \cdots U_n$ form a finite open covering of Y. Hence the space (Y, T_Y) is compact.

Theorem # A top-space X is Countably compact iff every infinite subset of X has a limit point in X

(First see next Theorem)

Proof # Suppose that A is any infinite subset of a Countably Compact space X. Then A has a Countably infinite subset

$$B = \{x_1, x_2, \cdots x_n \cdots\}$$

Suppose that A and therefore also B has no limit point in X

Consider the subsets

$$C_n = \{x_n, x_{n+1} \cdots\} \quad n = 1, 2, \cdots$$

Since the derived set of every $C_n : n = 1, 2 \cdots$ is empty, so C_n are closed. Also $\{C_n : n = 1, 2 \cdots\}$ satisfy the finite intersection property because

$$C_{n_1} \cap C_{n_2} \cap \cdots \cap C_{n_0} = C_{n_0} \neq \phi$$

Where $n_0 = \max(n_1, n_2, \cdots n_n)$

Now $\bigcap^{\infty} C_n = \phi$

because if a point x_p of B is in $\bigcap_{n=1}^{\infty} C_n$, then
$x_p \notin C_{p+1}$

But then a collection of closed sets with F.I.P has empty intersection and so X is not countably Compact, a contradiction. Hence B, therefore Also A, has a limit point in X.

Conversely Suppose that every infinite subset of X has a limit point in X. We are to prove that X is countably compact

Suppose that X is not countably compact and $\{G_n\}_{n\in N}$ be a countable open cover which has no finite subcover. This means that $\bigcup_{i=1}^{n} G_i$ does not contain X for any $n \in N$.

Let $F_n = \left(\bigcup_{i=1}^{n} G_i\right)'$

Then we have

$$F_1 = G_1'$$
$$F_2 = (G_1 \cup G_2)' = G_1' \cap G_2'$$
$$= F_1 \cap G_2' \subseteq F_1$$

$\Rightarrow$

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \text{-----}$.

i.e each F_n is non-empty closed set contained in the preceding one. From each F_n, we choose a point x_n and let

$$E = \bigcup_{n\in N} \{x_n\}$$

The set E can not be finite because there would be some point in an infinite number and hence all of the sets F_n and this would contradict the fact that family $\{G_n\}_{n\in N}$ is an open covering of X. Hence is infinite. and by hypothesis has a limit point x in X

$\because$ $\{G_n\}_{n\in N}$ is a covering of X

$\Rightarrow x$ is in G_{n_0} for some $n_0 \in N$

$\because x$ is a limit point of E

$\therefore G_{n_0} \cap E \setminus \{x\} \neq \phi$

$\Rightarrow$ G_{n_0} intersects with some F_n, other than x

But no element of $\{G_n\}_{n \in N}$ intersects with any element F_n, hence a contradiction. Thus X is countably compact

Theorem # Let X be a top-space. Then any infinite subset of X has a limit point iff every countable infinite subset of X has a limit point

Proof # Let every infinite subset of X has a limit point in X, then it immediately follows that every countable infinite subset of X also has a limit point in X

Conversely suppose that every countably infinite subset of X has a limit point in X. Let A be any infinite subset of X. Then A contains a countably infinite subset

$$B = \{x_1, x_2, \ldots x_n \ldots\}$$

By assumption B has a limit point x in X. But x is also a limit point of A

B.W Property: A top-space X is said to have B.W property if every sequence in X has at least one cluster point in X

OR

Equivalently X is said to have B.W property if every infinite subset of X has a limit point in X.

Comments # Since we have proved that a top-

space is countably compact iff every infinite subset it has a limit point in it, A Countably compact space satisfies B.W.P.

Theorem A top-space is Countably Compact iff has the B.W property OR

A top-space has B.W.P iff it is Countably Compact.

Proof #. We know that a top space X is Countably Compact iff every countable family of closed sets with F.I.P has a non-empty intersection.

Suppose that X has B.W property and $\{F_i : i \in I\}$ be a countable family of closed sets with finite intersection property.

Since the intersection

$$H_n = \bigcap_{k=1}^{n} F_k \text{ is empty for no } n$$

, therefore we may choose for each n an element $x_n \in H_n$

By B.W.P the sequence $\{x_n\}$ has a cluster point x. But

$$x_n \in F_i \quad \forall\, n \geq i$$

and so x must belong to F_i since F_i is closed. Thus $x \in F_i \ \forall i$ and x belongs to intersection of F_i i.e

$$\bigcap_{i \in I} F_i \neq \phi$$

$\Rightarrow$ X is Countably Compact.

Conversely suppose that X is Countably Compact and that $\{x_n\}$ is a sequence from X. Let B_n be the set $\{x_n, x_{n+1}, - - - - -\}$. Then $\{\overline{B}_n\}$ is a Countable collection of closed sets with F.I.P. Since X is Countably Compact, this collection

$\{\bar{B}_n\}$ has a non-empty intersection.

Let $x \in \bigcap \bar{B}_n$

The point x is a cluster point of the sequence, because given N and any open set O containing x we have $x \in \bar{B}_N$ and so there must be an $x_n \in O$ with $n \geq N$

Theorem # A Compact subset of a topological space is countably compact.

Proof # Since a subset compact with respective to a subspace of $(X, \mathcal{T})$ is compact iff it is compact relative to $(X, \mathcal{T})$, Therefore we may assume that the topological space X is compact and show that every infinite subset of X has a limit point in X. Suppose that E is an infinite set with no limit point. We know that each point $x \in X$ is not a limit pont of E, there must exist an open set G_x containing x such that $E \cap G_x \setminus \{x\} = \phi$

Clearly $E \cap G_x$ contains, at most, the one point point x itself. Since the family $\{G_x\}_{x \in X}$ forms an open covering of the compact space, there must be some infinite subcovering $X = \bigcup_{i=1}^{n} G_{x_i}$

Forom this it follows that $E = E \cap X = E \cap (\bigcup_{i=1}^{n} G_{x_i})$

$$= \bigcup_{i=1}^{n} (E \cap G_{x_i})$$

is a finite union of sets, each containing, at most, one element and so E is finite which is a contradiction. Thus every infinite subset of X must have at least one limit point and hence X is countably compact

Theorem# Let X be a T_1- space. Then X is Countably compact iff X satisfies the Bolzano- Weirstrass Property

Proof# Suppose that X is Countably Compact. Then X satisfies B.W property because every Countably Compact spaces satisfies B.W.P.

Conversely suppose that X is a T_1- space and satisfies the B.W.P. Let

$$\{C_1, C_2 - - - C_n, - - - \}$$

be any Countable collection of closed sets with F.I.P.

Consider the subsets

$$F_1 = C_1$$
$$F_2 = C_1 \cap C_2$$
$$- - - - -$$
$$F_n = C_1 \cap C_2 \cap C_3 \cap - - - \cap C_n$$

Then $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots \supseteq F_n - - -$

If the sequence $\{F_n\}$ Consists of a finite no of terms only, then for some natural number k,

$$F_k \subseteq \bigcap_{n=1}^{\infty} F_n \neq \phi$$

So suppose that the sequence $\{F_n\}$ Consists of an infinite no of terms. Then selecting points

$$x_1, x_2, - - - x_n - - -$$

such that $x_k \in F_k$, $k = 1, 2, \ldots$

we get an infinit set

$$A = \{x_1, x_2 \ldots\ldots x_n - - -\}$$

By B.W property A has a limit point x in X. clearly x is different from all x_i's

Let U be any open set containing x and n any natural number.

Since X is a T_1- space, $\exists$ an open set V Containing x such that

$$V \cap (\{x_1, x_2 \ldots\ldots x_{n-1}\} \setminus \{x\}) = \phi$$

Here we can take V as

$$V = V_1 \cap V_2 \cap V_3 \cap \ldots \cap V_n$$

where for each k, $x \in V_k$ and $x_k \notin V_k$, so that $x_1, x_2 \ldots\ldots x_k$ are not in V. Thus $U \cap V$ and hence also U is such that

$$U \cap \{x_n, x_{n+1} \ldots\ldots\} \setminus \{x\} = \phi$$

Since U is an arbitrary open set containing x, x is a limit point of

$$A_n = \{x_n, x_{n+2} \ldots\ldots\} \subseteq F_n$$

$\therefore$ F_n is closed, $x \in F_n$. Again since n is any natural number,

$$x \in \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} C_n \neq \phi$$

Thus X is Countably Compact.

Theorem # (a) Every compact space is Countably Compact, and also a Lindeloff space

(b) # A Countably compact Lindeloff space is Compact

Proof # (a) Let $(X.T)$ be a compact space. Since for every open covering of X, there exists a finite sub covering, the same is true for every Countable open covering. Hence $(X.T)$ is Countably Compact. Also since a finite cover is necessarily a Countable sub-covering, it follows that $(X.T)$ is also a Lindeloff space.

(b) # Let $(X.T)$ be a Countably Compact, Lindeloff space. Let C be an open covering of X. As $(X.T)$ is a Lindelooff space, there exists a Countable sub-covering C_1 of C for X. Again since

X is countably compact, for the countable open cover C_1 of X. there exists a finite sub-covering C_2. Then C_2 is finite sub-covering of C for X. Hence the space (X. T) is compact.

Sequentially Compact Space

A top-space X is said to be sequentially compact if every infinite sequence from x has a convergent subsequence in X

Sequentially Compact Sub-Set of a Space

A subset A of a top-space (X. T) is said to be sequentially compact iff every sequence in A has a sub-sequence convergent in A.

Theorem # (a) A sequentially compact space is countably compact.

(b) # Any countably compact, 1st countable space is sequentially compact

Proof #(a)# Let (X. T) be a sequently compact space and let $\langle x_1, x_2, x_3 \ldots \rangle$ be any infinite sequence in X. The the sequence $\{x_n\}$ has a cgt sub-sequence. The limit of the convergent sub-sequ is a cluster point for sequence $\{x_n\}$. Hence (X. T) is countably compact because a top-space is countably compact iff every infinite sequence in it has a cluster point. OR Sequentially Compactness $\Rightarrow$ B. W. P $\Rightarrow$ Countably Compactness.

(b) # Let (X. T) be a countably compact, 1st countable space. Let $\{x_n\}$ be an infinite sequence in X. Since X is countably compact,

, it is also Frechet Compact i.e every infinite sequence has an accumulation point in it. Hence the sequence $\{x_n\}$ has a accumulation point x (say) in X. Again since the space (X, T) is first Countable and a point x in 1st Countable space is an accummulation point of a subset E iff, there exists an infinite sequence of points $x_1, x_2 \ldots$ $(x_i \neq x)$ in E, such that $\lim_{n\to\infty} x_n = x$, therefore there exists a sub-sequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that

$$\lim x_{n_k} = x .$$

Thus the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$. Hence (X, T) is sequentially Compact.

Note In proving part (b) we have merely used the property of Frechet Compact (in place of 1st Countable Compactness). Hence every Frechet Compact, 1st Countable space is sequentially Compact.

Theorem # For second Countable, hence 1st Countable T_1-space, any one of the four properties (i) Compactness (ii) Countable Compactness (iii) sequential Compactness and (iv) Frechet Compactness, implies the other three.

Theorem # Every finite subset of a topological space is sequentially Compact.

Proof # Let (X, T) be a top-space and $A \subseteq X$ such that

$$O(A) = n \quad \text{i.e}$$

$$A = \{a_1, a_2, \ldots a_n\} \quad a_\alpha \in X, \alpha = 1, 2 \ldots n$$

Let $\{a_n\} = \langle a_1, a_2, a_3 \ldots a_n \ldots \rangle$ be a sequence in A.

Since A is finite, so some of the elements are

repeated infinitely in $\{a_n\}$. Suppose $a_0 \in A$ is repeated infinitely and we have

$$\{a_n\} = \{a_1, a_2, \ldots a_n, a_0, a_0, a_0 \ldots\}$$

Select the sub-sequence

$$\{b_n\} = \{a_2, a_3 \ldots a_n, a_0, a_0, a_0 \ldots\}$$

Let G be an open set containing a_0. Then G contains all but all but possibly finite terms of the sequence $\{b_n\}$. Hence a_0 is a limit point of $\{b_n\}$ and $a_0 \in A$.

Hence A is sequentially compact.

Problem # Show that $A = (0,1) \subseteq R$ is not sequentially compact.

Sol # Let us consider the sequence

$$\{a_n\} = \left\{\frac{1}{n+1}\right\} \quad n \in N$$

Then any subsequence of $\{a_n\}$ will converge to 0 but $0 \notin (0,1)$. Hence $A = (0,1)$ is not sequentially compact.

But $[0\ 1]$ is sequentially compact

Compactness In Metric Space

We shall shall show that notions of compactness, countable compactness, and sequential compactness coincide

B. W. Property # A metric space is said to have the B.W property if every infinite subset has a limit point

ϵ- Dense Subsets #

Let (X, d) be a metric space and $A \subseteq X$. Let ϵ be an arbitrary +ve real number. Then set A is

(i) is called ϵ- dense iff
$\forall x \in X$, $\forall a \in A$, $d(x,a) < \epsilon$

(ii) Set A is called ϵ-net of $(X.d)$ if for every $x \in X$, there is at least one $a \in A$ such that

$$d(a, x) < \epsilon$$

or equivalently $x \in B(a, \epsilon)$

Thus A is called an ϵ-net of (X,d) if every $x \in X$ is in at least one open ball of radius ϵ, centred at some point of A.

ϵ-Net for a Subset

Let $(X\ d)$ be a metric space and ϵ be any +ve real number. ϵ-net for a subset M of metric space is a finite set F of X such that for every $x \in M$, there is at least one $a \in F$ such that

$$d(x,d) < \epsilon \quad \text{or} \quad x \in B(a, \epsilon)$$

i.e

$$M \subseteq U\{ B(a,\epsilon) : a \in F\}$$

Example # Let $X = \{ (x,y) : x^2+y^2 \leq 1\}$ be the unit disk with centre at origin and radius 1.
Let $\epsilon = 1/2$. Then
$F = \{ (1,0), (0,1), (-1,0), (0,-1), (0,1/2), (0,0)$
$(1/2, 1/2), (1/2, -1/2), (-1/2, 0), (-1/2, 1/2), (0, -1/2)$
$(1/2, -1/2)\}$

is an ϵ-net for X
Similarly for $\epsilon = 3/4$

$F_1 = \{ (1,0), (0,1), (-1,0), (0,-1), (0,0)\}$
is an ϵ-net for X

Theorem # A metric space is sequentially compact iff it has B.W. property.

Proof # Let X be a metric space which is sequentially compact and A is an infinite subset of X. We are to show that A has a limit point in X. Since A is an infinite set, we can extract a sequence out of A. Let $\{a_n\}$ be a such sequence. Since X is sequentially compact, $\{a_n\}$ contains a subsequence $\{a_{n_k}\}$ such that

$$\{a_{n_k}\} \longrightarrow a_0 \in X$$

Let us consider the set of points of the subsequence $B = \{a_{n_1}, a_{n_2} \text{ ----- } a_{n_k} \ldots\}$. Then a_0 is a limit point B because if a convergent sequence in a metric space has infinitely many distinct points, its limit is a limit point of the set of points of the sequence. But $B \subseteq A$, so $a_0 \in X$ and a_0 is also a limit point of A. Hence B.W. property is satisfied.

Conversely let B.W property is satisfied in a metric space X. We prove that X is sequentially compact. Let $\{a_n\}$ be an arbitrary sequence. Then we have following cases

Case I # If sequence $\{a_n\}$ has a point which is infinitely repeated, then it has a constant subsequence and this subsequence is clearly convergent.

Case II # If no point of $\{a_n\}$ is infinitely repeated, then the set A of points of this sequence is infinite. By B.W property this set has a limit point 'a' in X. So we can extract a subsequence out of A which has 'a' as limit point.

Thus X is sequentially compact space.

Comments # The terms limit and limit point are often confusing for those who are not thoroughly accustomed to them. e.g on the real line the constant sequence $\{1, 1, 1, \ldots 1 \ldots\}$ is convergt with limit 1; but the set of points of this sequence is the set $\{1\}$ and 1 is not limit point of this set because if it is, then every open sphere centred at 1 must contain an infinite no of distinct points of the set $\{1\}$. Thus a sequence of points in set is not a subset of the set: it is a function defined on +ve integers with values (of) in the set and is usually specified by listing its values. A sequence may have a limit, but can not have a limit point and the set of points of sequence may have a limit point but can not have a limit

Examples #

1) # Every finite metric space (X, d) is compact

Proof # Let C be an open cover of a finite set $X = \{x_1, x_2, x_3 \ldots \ldots x_n\}$ for each $1 \le i \le n$. choose a set $G_i \in C$ with $x_i \in G_i$. Then the finite collection

$\{G_1, G_2 \ldots G_n\} \subseteq C$ is also a cover for X because.

$x_i \in G_i \quad \forall i = 1, 2 \ldots n$

$\Rightarrow \bigcup_{i=1}^{n} \{x_i\} \subseteq \bigcup_{i=1}^{n} G_i$

$\Rightarrow X \subseteq \bigcup_{i=1}^{n} G_i$

But $\bigcup_{i=1}^{n} G_i \subseteq X$

$\Rightarrow X = \bigcup G_i$

1) Note It follows that every finite subset A of X is also compact

2) Finite discrete metric space is compact, infinite discrete metric space is not compact. A discrete space X compact iff X is finite

Proof # Note Let X be any non-empty set define $d : X \times X \to R$ by

$$d(x,y) = 0 \quad x = y$$
$$= 1 \quad x \neq y$$

Then d is metric on X and is called trivial metric or discrete metric

Nessacity # Suppose X is discrete & infinite Then the infinite collection $C = \{\{x\} : x \in X\}$ is an open cover of X, but no subcollection of C is a cover of X except X itself. Thus it is necessary that X be finite

(ii) Sufficiency If X is finite, then it is necessarily compact by example (1)

3) The subsets $A =]0\ 1]$ and $B = \mathbb{Z}$ are not compact in (R, d).

Note A metric space is compact iff it is sequentially compact.

(i) The sequence $\{1/n\}$ is convergent to $0 \notin A$, so all subsequences of $\{1/n\}$ converge to $0 \notin A$. Thus no subsequence is convergent in A. So A is not sequentially compact and hence is not compact

(ii) Then sequence $\{n\}$ in $\mathbb{Z}$ is unbounded. Thus no subsequence is cgt in B, so B is not compact.

Theorem # Every compact metric space has the Bolzano - Weirstrass Property (B.W.P) i.e every infinite subset of a compact metric space has a limit point.

Proof # Let (X,d) be a compact metric space and A be an infinite subset of X. Assume that A has no limit point. Then each point of X is not a limit point of A, so each point of X is the centre of an open sphere (open set) which contains no point of A different from its centre. Thus for any point x in X, there exists an open set G_x, that contains not point of A except possibly x, as only $x \in G_x$

Collection $C = \{G_x : x \in X\}$ is an open cover for X. Since X is compact, C admits of a finite sub-cover $\{G_{x_i} : i = 1, 2 \dots n\}$ such that

$$X = \bigcup_{i=1}^{n} G_{x_i}$$

$\because A \subseteq X$

$\therefore A \subseteq \bigcup_{i=1}^{n} G_{x_i}$

Since each G_{x_i} contains at most one point of A, A must be finite set. This contradicts the fact that A is infinite. Consequently A must have a limit point. Thus X has B.W.P.

Theorem # A finite subset $F = \{a_1, a_2 \dots a_n\}$ is an ϵ-net for X iff

$$X = \bigcup_{i=1}^{n} B(a_i, \epsilon)$$

Proof # Suppose that $F = \{a_1, a_2 \dots a_n\}$ is an ϵ-net for X. Then for each $x \in X$, there is an $a_i \in F$ such that

$$x \in B(a_i, \epsilon)$$

Thus $X = \bigcup_{x \in X} \{x\} \subseteq \bigcup_{i=1}^{n} B(a_i, \epsilon) \subseteq X$

$$\Rightarrow X = \bigcup_{i=1}^{n} B(a_i, \epsilon)$$

Conversely suppose that

$$X = \bigcup_{i=1}^{n} B(a_i\ \epsilon)$$

Let $x \in X$.

Then $x \in B(a_i\ \epsilon)$ for some i

i.e for any $x \in X$, there is an $a_i \in F$ such that

$$d(x\ a_i) < \epsilon$$

Hence F is an ϵ-net for X

Theorem # Every Compact subset of a metric space is closed

Proof Let A be any compact subset of a metric space X.

We show that $A^c = X \setminus A$ is open

Let $x \in A$ and $y \in A^c$. Then.

$$x \neq y$$

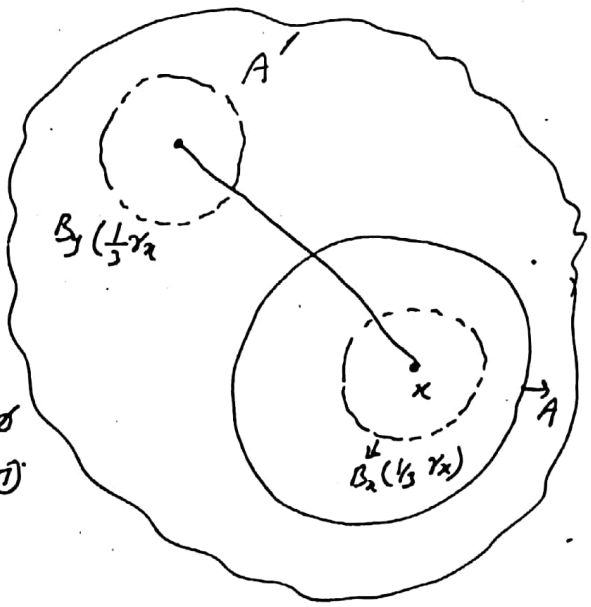
Suppose that

$$d(x,y) = \delta_x$$

The open balls

$B(x : \frac{1}{3}r_x)$ & $B(y : \frac{1}{3}r_x)$ are mutually disjoint i.e

$$B(x : \tfrac{1}{3}r_x) \cap B(y : \tfrac{1}{3}r_x) = \emptyset \longrightarrow (1)$$



If not let $z \in B_x \cap B_y$

Then

$$d(z,x) < \tfrac{1}{3}r_x \qquad d(z,y) < \tfrac{1}{3}r_x$$

$$d(x,y) \leq d(x,z) + d(z,y) \Rightarrow \delta_x \leq \tfrac{1}{3}r_x + \tfrac{1}{3}r_x$$

which is a contradiction

Let $C = \{B(x: \frac{1}{3}r_x) : x \in A\}$ be an open cover for set A i.e.

$$A \subseteq \bigcup \{B(x: \tfrac{1}{3}r_x) : x \in A\}$$

$\because$ A is compact.

$\therefore$ There exists a finite open cover C_1 of A given by

$$C_1 = \{B(x: \tfrac{1}{2}r_{x_i}) : 1 \le i \le n\}$$

Then $A \subseteq \bigcup_{i=1}^{n} \{B(x; \frac{1}{3}r_{x_i})\}$

$$A \subseteq H_x \qquad H_x = \bigcup_{i=1}^{n} B(x: \tfrac{1}{3}r_{x_i}) \rightarrow (2)$$

obviously, H_x is an open set. Let

$$C_y = \bigcap_{i=1}^{n} B(y: \tfrac{1}{3}r_{x_i})$$

Then C_y being the intersection of open balls containing y is an open set.

Now by (1)

$$B(x: \tfrac{1}{3}r_{x_i}) \cap B(y: \tfrac{1}{3}r_{x_i}) = \phi \quad 1 \le i \le n$$

$$\Rightarrow \quad B(x: \tfrac{1}{3}r_{x_i}) \cap C_y = \phi \quad [\because C_y \subseteq B(y: \tfrac{1}{3}r_{x_i})]$$

So $\left\{\bigcup_{i=1}^{n} B(x: \tfrac{1}{3}r_{x_i})\right\} \cap C_y = \phi$

$$\Rightarrow \quad H_x \cap C_y = \phi \qquad \rightarrow (3)$$

By (2) and (3) we obtain

$$A \cap C_y = \phi$$

$$\Rightarrow C_y \subseteq A'$$

Since $\bigcup \{C_y : y \in A'\} = A'$ and each C_y is open, it follows that A' is open and hence A is closed.

Theorem# Let K be a compact subset of metric space X and let $p \in X \setminus K$. Then there exist open sets G and H such that

$$p \in G \qquad K \subseteq H \qquad G \cap H = \phi \qquad G \subseteq K'$$

i.e Any compact set K and any point p not in K are separated by open sets

Proof# Let X be a metric space and K be its compact subset.

Since every metric space is Hausdorff space, X is Hausdorff.

$\Rightarrow$ For any $x \in K \qquad p \in K'$, there exists open balls H_x and G_p about x and p such that

$$H_x \cap G_p = \phi \longrightarrow (1)$$

Then $\{H_x : x \in K\}$ is clearly an open cover of set K. Since K is compact, this cover is reducible to a finite subcover $\{H_{x_i} : x_i \in K,\ 1 \le i \le n\}$ such that

$$K \subseteq \bigcup_{i=1}^{n} H_{x_i}$$

or $K \subseteq H \qquad \left[H = \bigcup_{i=1}^{n} H_{x_i},\ x_i \in K\right] \longrightarrow (2)$

Let $G = \bigcap_{i=1}^{n} G_{p_i}$, where G_{p_i} are open sets

Corresponding to open sets H_{x_i}.

According to (1)

$$H_{x_i} \cap G_{p_i} = \phi \qquad x_i \in K,\ p_i \in K',\ 1 \le i \le n$$

This shows that

$$G_{p_i} \subseteq H_{x_i}'$$

$$\Rightarrow \bigcap_{i=1}^{n} G_{p_i} \subseteq \bigcap_{i=1}^{n} H_{x_i}' = \left(\bigcup_{i=1}^{n} H_{x_i}\right)' = H'$$

$$\Rightarrow G \subseteq H' \subseteq K' \qquad \text{By (2) } K \subseteq H \Rightarrow H' \subseteq K'$$

$\Rightarrow G \cap H = \phi$. G, H are open sets, $p \in G$, $K \subseteq H$

Cor # Let K_1 and K_2 be disjoint compact subsets of a metric space X. Then, there exist disjoint open sets G & H such that

$$K_1 \subseteq G \quad , \quad K_2 \subseteq H$$

Proof # Let $x \in K_1$, then $x \notin K_2$ since $K_1 \cap K_2 = \phi$

Since K_2 is compact, $x \notin K_2$, hence by above theorem there exist open sets G_x & H_x, such that

$$x \in G_x \quad , \quad K_2 \subseteq H_x \quad \text{and} \quad G_x \cap H_x = \phi$$

Then $C = \{G_x : x \in K_1\}$ is an open cover of K_1. Since K_1 is compact, C admits of a finite subcover $C_0 = \{G_{x_1}, G_{x_2}, \ldots G_{x_n}\}$ such that

$$K_1 \subseteq \bigcup_{i=1}^{n} G_{x_i} \quad \text{or} \quad K_1 \subseteq G \quad \left[G = \bigcup_{i=1}^{n} G_{x_i} \right.$$

Let $H_{x_1}, H_{x_2}, \ldots H_{x_n}$ be corresponding open sets containing K_2 and

$$H = \bigcap_{i=1}^{n} H_{x_i}$$

Then $K_2 \subseteq H$ because $K_2 \subseteq H_{x_i}$ $1 \leq i \leq n$

obviously G & H are open sets. We need to prove that

$$G \cap H = \phi$$

Let $y \in G$. Then

$$y \in G \Rightarrow y \in G_{x_i} \text{ for some } x_i$$
$$\Rightarrow y \notin H_{x_i} \quad [G_x \cap H_x = \phi]$$
$$\Rightarrow y \notin \bigcap_{i=1}^{n} H_{x_i} = H$$

Thus $y \in G \Rightarrow y \notin H$, hence $G \cap H = \phi$

Theorem # Any closed subset (subspace) of a compact metric space is compact.

Proof # Let (X, d) be a compact metric space and F be a closed subset of X.

Let $C = \{G_\alpha : \alpha \in I\}$ be an open covering of F. Then

$$F \subseteq \bigcup_{\alpha \in I} G_\alpha$$

$$\therefore X = F \cup F' \subseteq \{\bigcup_{\alpha \in I} G_\alpha\} \cup F'$$

$\Rightarrow$ That $\{C, F'\}$ is also an open cover of X.

Since X is compact and F' does not cover any part of F, it follows that there exists a finite sub-cover $C_0 = \{G_{\alpha_1}, G_{\alpha_2} \ldots G_{\alpha_n}\}$ such that

$$X = \{\bigcup_{i=1}^{n} G_{\alpha_i}\} \cup F' = H \cup F' \longrightarrow (1) \qquad \left[H = \bigcup_{i=1}^{n} G_{\alpha_i}\right]$$

Complementing

$$\emptyset = H' \cap F$$

$$\Rightarrow F \subseteq (H')' = H \longrightarrow (2) \quad [(H')' = H]$$

By (*) & (2)

$$F \subseteq H \left(= \bigcup_{i=1}^{n} G_{\alpha_i}\right)$$

Thus the open cover C of F is reducible to a finite subcover C_0. Hence F must be compact.

Compactness in Sub Spaces

Theorem # Let (Y, d) be a subspace of metric space (X, d) and $A \subseteq Y$. Prove that A is compact in Y iff A is compact in X

Proof # Let A be compact in Y and $C = \{G_\alpha : \alpha \in I\}$ be an open cover of A in space X. Then

$$A \subseteq \bigcup_{\alpha \in I} G_\alpha$$

Now

$$A = A \cap Y \subseteq (\cup_\alpha G_\alpha) \cap Y = \cup_\alpha (G_\alpha \cap Y) \longrightarrow (1)$$

$\Rightarrow \{ G_\alpha \cap Y : \alpha \in I \}$ is an open cover of A in Y. Since A is compact in Y, this cover has a finite sub-cover $\{ G_{\alpha_i} \cap Y : i = 1, 2, \ldots n \}$ such that

$$A \subseteq \bigcup_{i=1}^{n} (G_{\alpha_i} \cap Y) \subseteq \bigcup_{i=1}^{n} G_{\alpha_i} \quad [G_\alpha \cap Y \subseteq G_\alpha]$$

$\Rightarrow C_0 = \{ G_{\alpha_i} : i = 1, 2 \ldots n \}$ is a finite sub-cover of C in space X. It follows that A is compact in space X.

Conversely Let A be compact in X. Let $\{ H_\beta : \beta \in I \}$ be an open covering of A in $\mathcal{U}(Y)$. Then for each β, there is an open set G_β in X such that

$$H_\beta = Y \cap G_\beta \qquad (Y \text{ is subspace of } X)$$

Now

$$A \subseteq \cup_\beta H_\beta = \cup_\beta (Y \cap G_\beta)$$
$$= Y \cap \{ \cup_\beta G_\beta \} \subseteq \cup_\beta G_\beta$$
$$[Y \cap A \subseteq A]$$

$\Rightarrow \{ G_\beta : \beta \in I \}$ is an open covering of A in space X. Since A is compact in X, this covering must admit of a finite subcover $\{ G_{\beta_i} : i = 1, 2 \ldots m \}$ such that

$$A \subseteq \bigcup_{i=1}^{m} G_{\beta_i}$$

Now $$A = Y \cap A \subseteq Y \cap \{ \bigcup_{i=1}^{m} G_{\beta_i} \}$$
$$= \bigcup_{i=1}^{m} (Y \cap G_{\beta_i})$$
$$= \bigcup_{i=1}^{m} H_{\beta_i}$$

where each H_{β_i} is an open in Y. Hence $\{H_{\beta_i} : i = 1, \dots n\}$ is a finite sub-cover in space Y. It follows that A is compact in space Y.

Comment#. Compactness is an intrinsic (belonging to or part of the real nature of sth/sb) property of sets.

Theorem# Let (X, d) be a metric space and $A \subseteq X$. Then A is compact iff every covering of A by open subsets of X has a finite subcovering.

Proof# Let A be compact. This means A is compact as a subspace of X. Let $\{G_\alpha : \alpha \in I\}$ be a covering of A by open subsets of X. We must finite a finite subset $I_1 \subseteq I$, such that $\{G_\alpha : \alpha \in I_1\}$ is a covering of A.

Now $\{G_\alpha \cap A : \alpha \in I\}$ is an open covering of A by open subsets of A. Since A is compact, there is a finite subcovering $\{G_\alpha \cap A : \alpha \in I_1 \subseteq I\}$ such that

$$A \subseteq \bigcup_{\alpha \in I_1} (G_\alpha \cap A) \subseteq \bigcup_{\alpha \in I_1} G_\alpha$$

Hence $\{G_\alpha : \alpha \in I_1\}$ is the required subcovering

Conversely let every covering of A by open subsets of X have a finite subcovering. Let $\{H_\alpha : \alpha \in I\}$ be a covering of A by open subsets of A. By subspace-criterion for open sets, each H_α has the form

$$H_\alpha = G_\alpha \cap A \quad \text{where } G_\alpha \text{ is open in } X$$

So

$$A \subseteq \bigcup_{\alpha \in I} H_\alpha = \bigcup_{\alpha \in I} (G_\alpha \cap A)$$

$$= \left(\bigcup_{\alpha \in I} G_\alpha\right) \cap A \subseteq \bigcup_{\alpha \in I} G_\alpha$$

$\Rightarrow \{G_\alpha : \alpha \in I\}$ is a cover for A in X

Since every open covering of A in X has a finite sub-cover

; therefore there is a finite sub-cover $\{G_{\alpha_i} : i=1,2,\dots n\}$ such that

$$A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

$$\Rightarrow A \cap A \subseteq \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap A$$

$$A \subseteq \bigcup_{i=1}^{n} (G_{\alpha_i} \cap A)$$

$$= \bigcup_{i=1}^{n} H_{\alpha_i} \qquad H_{\alpha_i} = G_{\alpha_i} \cap A$$

This proves that A is compact.

Theorem # A metric space (X, d) is compact iff every basic open covering of X is reducible to a finite sub-covering

Proof # Let (X, d) be a compact metric space and β be a base for the system of open subsets of X.

Since X is compact, every open covering of admits of a finite subcovering and hence every basic open cover has also a finite sub-cover.

Conversely let $\beta = \{G_\sigma : \sigma \in I_\sigma\}$ be a base for open subsets of X

let every basic open covering has a finite sub-covering. We prove that (X, d) is compact.

Let $C = \{G_\alpha : \alpha \in I\}$ where $I \supseteq I_\sigma$, be an open covering of X. Since β is base, therefore

$$G_\alpha = \bigcup_{\sigma \in I_\alpha} G_\sigma, \quad \text{for each } \alpha$$

Consequently totality of all such members of β is a basic open covering of X. By assumption, this covering is reducible to a finite sub-covering say $\{B_1, B_2, \dots B_n\}$

Such that

$$X = \bigcup_{i=1}^{n} B_{\alpha_i}$$

for each B_{α_i}, we can find a G_{α_i} from C such that

$$B_{\alpha_i} \subseteq G_{\alpha_i}$$

But then this amounts to $\{G_{\alpha_i} : i = 1, 2 \ldots n\}$ is a finite subcovering of C. Consequently, (X d) is a compact metric space.

Free and fixed family of subsets

A family of subsets $\{F_\alpha : \alpha \in I\}$ of a space is called free if $\bigcap_\alpha F_\alpha = \phi$ and is called fixed if

$$\bigcap_\alpha F_\alpha \neq \phi$$

Theorem # A metric space (X.d) is compact iff every family of closed subsets of X with F.I.P has a non-empty intersection.

Proof # Let (X.d) be compact and $\{F_\alpha\}$ be a family of closed subsets of X with F.I.P. we show that

$$\bigcap_\alpha F_\alpha \neq \phi$$

If possible let $\bigcap_\alpha F_\alpha = \phi$

$$\left(\bigcap_\alpha F_\alpha\right)' = \phi'$$

$$\bigcup_\alpha F_\alpha' = X$$

$\Rightarrow \{F_\alpha' : \alpha \in I\}$ is an open cover of X. Since X is compact, this cover has a finite subcover $\{F_{\alpha_i}' : i = 1, 1 \ldots n\}$ such that

$$X = \bigcup_{i=1}^{n} F_{\alpha_i}'$$

By complementing

$$\bigcap_{i=1}^{n} F_{\alpha_i} = \phi$$

which contradicts the F.I.P of $\{F_\alpha\}$. Hence

$$\bigcap_{\alpha} F_\alpha \neq \phi$$

Conversely let every collection of closed subsets of X with F.I.P have non-empty intersection.

Let $\{G_\alpha : \alpha \in I\}$ be an open covering of X: so that

$$X = \bigcup_{\alpha} G_\alpha$$

By complementing

$$\bigcap_{\alpha} G_\alpha' = \phi$$

$\Rightarrow \{G_\alpha' : \alpha \in I\}$ is the collection of closed subsets with empty intersection. By assumption this family does not have F.I.P. As such there is a finite sub-collection $\{G_\alpha' : \alpha \in I\}$, where I is finite or $\{G_\alpha' : \alpha = 1, 2, \cdots m\}$ such that

$$\bigcap_{i=1}^{m} G_\alpha' = \phi$$

By complementing

$$\bigcup_{i=1}^{m} G_\alpha = X$$

$\Rightarrow$ Covering $\{G_\alpha : \alpha \in I\}$ has a finite subcovering. Hence (X, d) must be compact.

Bounded Set in a Metric Space

A subset E of a metric space (X, d) is said to be bounded iff there exists a real number β such that

$$d(x, y) < \beta \quad \forall x, y \in E$$

i.e $\delta(E) < \infty$ $\quad$ $\delta(E)$ - diameter of E

Totally Bounded Set OR Pre Compact Set

A subset E of a metric space is totally bounded or precompact iff it has an ϵ-net for every +ve real number ϵ

#Totally Bounded Space#

A metric space X is called totally bounded iff it has an ϵ-net for every +ve real number ϵ i.e for each $\epsilon > 0$ there is a finite collection of spheres of radius ϵ which covers X

Comments # Since an ϵ-net is a finite set of points, it is clear every totally bounded set is bounded.

Since every Compact space is Countably Compact, therefore to prove that every Compact metric space is totally bounded we need only prove that every Cerntably Compact metric space is totally bounded.

Theorem # Every Countably Compact metric space is totally bounded.

Proof # Suppose that the Countably Compact metric space (X, d) is not totally bounded. Then for some fixed +ve number ϵ, X must have no ϵ-net.

Let $x_1 \in X$. Then the finite set $\{x_1\}$ is not an ϵ-net for X i.e $X \neq B(x_1, \epsilon)$. So there exist a point $x_2 \notin B(x_1, \epsilon)$ i.e $d(x_1, x_2) \geq \epsilon$

Now the finite set $\{x_1, x_2\}$ is also not an ϵ-net for X i.e $X \neq \bigcup_{i=1}^{2} B(x_i, \epsilon)$

So there exists a point $x_2 \notin \bigcup_{i=1}^{2} B(x_i, \epsilon)$ i.e $d(x_1, x_2) \geq \epsilon$ and $d(x_2, x_3) \geq \epsilon$

We now proceed by induction. If we have defined a set of points $\{x_1, x_2, \ldots, x_n\}$ such that

$$d(x_i, x_j) \geq \epsilon \quad \text{whenever } i \neq j,$$

then this finite set is not ϵ-net for X and so there exists a point $x_{n+1} \notin \bigcup_{i=1}^{n} B(x_i, \epsilon)$ i.e $x_{n+1} \in X \setminus \bigcup_{i=1}^{n} B(x_i, \epsilon)$

with the property $d(x_i, x_j) \geq \epsilon$ $i \neq j$, $j = 1, 2, \ldots, n+1$.

Consider now the sequence $\{x_n\}$ constructed in this way. This sequence has distinct terms.

Let $E = \bigcup_{n \in N} \{x_n\}$.

By Countable compactness, E must have a limit point $x \in X$. But by definition of limit point $B(x, \epsilon/2)$ must contain an infinite number of points of E.

Let $x_m, x_n \in B(x, \epsilon/2)$

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

which contradicts the fact that

$$d(x_m, x_n) \geq \epsilon$$

Hence X is totally bounded.

Theorem # Every totally bounded metric space is bounded

Proof # Let (X, d) be a total bounded metric. Then for each $\epsilon > 0$ there is an ϵ-net $F = \{a_1, a_2, \ldots a_n\}$ for (X, d) i.e

$$X = \bigcup_{i=1}^{n} B(a_i, \epsilon)$$

$\Rightarrow$ for each $x \in X$, there is an $a_i \in F$ such that $d(x, a_i) < \epsilon$ $i = 1, \ldots, n$ $(n \geq 1)$ for some a_i

Similarly for a $y \in X$, There is an $a_j \in F$ such that $d(y, a_j) < \epsilon$ Hence from triangle inequality

$$d(x,y) \leq d(x, a_i) + d(a_i, y) + d(a_i, a_j)$$
$$\leq d(a_i, a_j) + 2\epsilon$$

So $$\delta(X) = \sup_{x,y \in X} d(x,y) \leq \sup_{a_i, a_j \in F} d(a_i, a_j) + 2\epsilon$$
$$\leq \delta(A) + 2\epsilon$$

So $\delta(X)$ is finite. Therefore X is bounded.

Comments # The converse of this is not true

e.g

Let (X, d) be any metric. Then (X, d') where d' is defined by

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)} \qquad x, y \in X$$

is also a metric space.

$\because \forall\, x, y \in X$

$d'(x,y) < 1$ in (X, d')

Therefore $\delta(X) = \sup_{x,y \in X} d'(x,y) \leq 1$

Hence (X, d') is bounded.

Let $X = R$

Define $d' : X \times X \longrightarrow R$ by

$$d(x,y) = \frac{|x-y|}{1 + |x-y|} \qquad x, y \in X$$

Then whenever x, y are integers

$$d'(x,y) \geq 1/2$$

Therefore There is no ϵ-net for $\epsilon = 1/2$. Hence (X, d') is not totally bounded.

The real line (R, d) with usual metric is neither bounded nor totally bounded. With $d(x,y) = \min\{|x-y|, 1\}$, R is bounded but not totally bounded.

because when x, y are integers

$$d_1(x, y) \geq 1$$

So there is no ϵ-net for $\epsilon = 1$. Hence (R, d_1) is not totally bounded.

Theorem # Every totally bounded metric space (X, d) is separable

Proof # Since (X, d) is totally bounded metric space, for each $n \in N$, $\exists$ an ϵ-net, E_n with $\epsilon = 1/n$. Trivially $E = \cup \{E_n : n \in N\}$ is countable.

For any given $x \in X$

$$d(x, E) \leq d(x, E_n) < \frac{1}{n} \quad \forall n \in N$$

$$\Rightarrow \quad d(x, E) = 0$$

$$\Rightarrow \quad x \in \bar{E} \quad \Rightarrow \quad X \subseteq \bar{E} \quad \text{but} \quad \bar{E} \subseteq X$$

Hence $\bar{E} = X$

$\Rightarrow$ E is everywhere dense in X, so X has a countable dense subset. Thus X is separable.

Comments # Compact metric spaces are always separable

Theorem # A metric space X is compact iff it is both complete and totally bounded.

Proof # If X is compact, it is trivially bounded. If $\{x_n\}$ is a Cauchy sequence in X, then $\{x_n\}$ must have a cluster point. But a Cauchy sequence which has a cluster point converges to the cluster point. Thus X is complete.

Conversely suppose that X is complete and totally bounded. To show that X is compact it is sufficient to show that every infinite sequence $\{x_n\}$ has a convergent subsequence. Since X is totally bounded

, we may cover X by a finite number of spheres of radius 1. Among these spheres there must be a sphere S_1 which in must contain infinitely many terms of the sequence $\{x_n\}$. Covering X by a finite no of spheres of radius $1/2$, we can find among them a sphere S_2 s.t. $S_1 \cap S_2$ contains infinitely many terms of the sequence $\{x_n\}$. Continuing, we obtain a sequence $\{S_k\}$ having radius $1/k$ such that $S_1 \cap S_2 \cap S_3 \cap S_4 \cap \dots \cap S_k$ contains infinitely many terms of the sequence $\{x_n\}$. Since there are infinitely many terms of the sequence in $S_1 \cap S_2 \cap S_3 \cap \dots \cap S_k$, we may choose n_k so that $n_k > n_{k-1}$ and $x_{n_k} \in S_1 \cap S_2 \cap S_3 \cap \dots \cap S_k$. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and it must be a Cauchy sequence, since

$$d(x_{n_k}, x_{n_l}) \leq \frac{2}{N} \quad \text{for } k, l \geq N.$$

Since X is complete, this sequence converges.

Theorem # (Lemma) Let (X, d) be a totally bounded metric space and $A \subseteq X$ be an infinite subset. The for every $\epsilon > 0$, there exists an infinite set $C \subseteq A$ with $\delta(C) < \epsilon$

Proof # Here A is an infinite set and $\epsilon > 0$ is given. Let $E = \{x_1, x_2, \dots, x_n\}$ be an $\epsilon/3$ net in (X, d). Then

$$X = \bigcup_{k=1}^{n} B(x_k : \epsilon/3)$$

$$A = A \cap X = \bigcup_{k=1}^{n} \{A \cap B(x_k : \tfrac{1}{3}\epsilon)\}$$

Thus at least one of the sets $A \cap B(x_k : \frac{1}{3}\epsilon)$ must be infinite. Call that set by name C

Then $C = A \cap B(x_p : \frac{1}{3}\epsilon)$

$\Rightarrow C \subseteq A$ if $\delta(C) < \epsilon$

Theorem # A metric space (X, d) is totally bounded iff each sequence $\{x_n\}$ in X has a Cauchy subsequence.

Proof # Let X be totally bounded and $\{x_n\}$ be any sequence in X. If the set $A = \{x_n : n \in \mathbb{N}\}$ is finite, then $\{x_n\}$ has a constant subsequence which is trivially a Cauchy subsequence. So we assume that A is infinite. Then by above lemma there exists an infinite subset $C_1 \subseteq A$ with $\delta(C_1) < 1$ (1-net). Choose n_1 such that $x_{n_1} \in C_1$ and by lemma there exists an infinite set $C_2 \subseteq C_1$ with $\delta(C_2) < 1/2$. Choose $n_2 > n_1$ such that $x_{n_2} \in C_2$. Proceeding continuously, we have chosen infinite subsets $C_k \subseteq C_{k-1} \subseteq \cdots C_2 \subseteq C_1 \subseteq A$ with $\delta(C_i) < \frac{1}{i}$, for $i = 1, 2, \cdots k$ and positive integers $n_k > n_{k-1} > \cdots > n_2 > n_1$ with $x_{n_i} \in C_i$. Now let C_{k+1} be an infinite subset of C_k with $\delta(C_{k+1}) < \frac{1}{k+1}$. Choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in C_{k-1}$. In this manner we may extract a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$

We now prove that $\{x_{n_k}\}$ is Cauchy. Let $\epsilon > 0$ be given. Choose k_0 so large that $k_0 > 1/\epsilon$. By our construction $x_{n_p} \in C_{k_0}$ $\forall\ p \geq k_0$. Thus if $p, q > k_0$, then $d(x_{n_p}, x_{n_q}) < \frac{1}{k_0} < \epsilon$. This proves that each sequence in (X, d) has a Cauchy subsequence.

Converse Let (X, d) not be totally bounded. Then there is an $\epsilon > 0$ for which X contains no ϵ-net. Let $x_1 \in X$ be arbitrary: choose $x_2 \in X$ such that $d(x_2, x_1) \geq \epsilon$. This is possible because $\{x_1\}$ can not be an ϵ-net. Choose $x_3 \in X$ such that $d(x_3, x_1) \geq \epsilon$ & $d(x_3, x_2) \geq \epsilon$. This choice is possible since $\{x_1, x_2\}$ is not ϵ-net. Continued in this way we obtain $E = \{x_1, x_2, \cdots x_k\}$ such that $d(x_i, x_j) \geq \epsilon$ $\forall\ i \neq j$ and $i, j = 1, 2, \ldots k$

and note that E can not be ϵ-net in X. Thus there is an $x_{k+1} \in X$ with $d(x_{k+1}, x_i) \geq \epsilon$ for $i = 1, 2, \ldots k$. Obviously, the sequence $\{x_n\}$ has no cauchy sub-sequence. This contradiction proves that metric space (X, d) is totally bounded.

OR

Suppose that X is totally bounded and $\{x_n\}$ is a sequence of points in X. Since X contains an ϵ-net for $\epsilon = 1/2$, it is contained in the union of a finite number of balls of radius $1/2$. One of these balls must contain a subsequence $\{x_n^{(1)}\}$ of the origional sequence and the distance between any two points of this subsequence is less than 1. Now with $\epsilon = 1/4$, we obtain a subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ in which the distance between any two points of this subsequence is less than $1/2$. Proceeding by induction, for $\epsilon = \frac{1}{2k}$, we obtain a subsequence $\{x_n^{(k)}\}$ of $\{x_n^{(k-1)}\}$ in which the distance between any two points is less than $1/k$. We assert that the diagonal sequence $\{x_n^{(n)}\}$ is a Cauchy subsequence of $\{x_n\}$. Since for any $\epsilon > 0$ we may choose $N(\epsilon) > \frac{1}{\epsilon}$ and then $d(x_m^{(m)}, x_n^{(n)}) < \epsilon$ whenever $m, n > N(\epsilon)$. Conversely, if X is not totally bounded, there exists an $\epsilon > 0$ such that X has no ϵ-net. Let x_1 be any point of X. Then the finite set $\{x_1\}$ is not an ϵ-net for X and so there must exist a point $x_2 \notin B(x_1; \epsilon)$ such that $d(x_1, x_2) \geq \epsilon$. Now the finite set $\{x_1, x_2\}$ is also not an ϵ-net for X, so there exists a point $x_3 \notin \bigcup_{i=1}^{2} B(x_i, \epsilon)$ such that $d(x_1, x_3) \geq \epsilon$ and $d(x_2, x_3) \geq \epsilon$. If we have obtained a set of points $\{x_1, x_2, \ldots x_n\}$ such that

$d(x_i, x_j) \geq \epsilon$ whenever $i \neq j$. Then this finite set is not an ϵ-net for X and so there exists a point $x_{n+1} \notin \bigcup_{i=1}^{n} B(x_i, \epsilon)$ i.e $d(x_i, x_j) \geq \epsilon$ whenever $i \neq j$.

It is cleare that this sequence $\{x_n\}$ has no Cauchy subsequence. Hence X is totally bounded.

Lemma 1# If X is a compact metric space, then every sequence in X has a convergent subsequence.

Proof# Seppose that there exists a sequence $\{x_n\}$ in X which has no convergent subsequence. We claim that for every $x \in X$, there exists an open ball $B(x:\epsilon)$ such that $\{x_n \in B(x:\epsilon) : n \in P\}$ is finite. For if some $x \in X$ has the property that every open ball $B(x:\epsilon)$ contains x_n for infinitely many +ve integers n, we may produce a convergent subsequence of $\{x_n\}$ as follows. Choose $x_{n_1} \in B_1(x)$ Choose $x_{n_2} \in B_{1/2}(x)$ such that $n_2 > n_1$. Having chosen $x_{n_k} \in B_{1/k}(x)$, choose $x_{n_{k+1}} \in B_{1/k+1}(x)$ such that $n_{k+1} > n_k$. Then $\lim_{k\to\infty} x_{n_k} = x$ and hence $\{x_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence of $\{x_n\}$. This is a contradiction and hence for every $x \in X$, there

Lemma#2 Let X be a metric space in which every sequence has a convergent subsequence. Let $\epsilon > 0$. Then there exist $x_1, x_2, \ldots, x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} B(x_i : \epsilon)$$

Proof# Let d be metric on X. Let $x_1 \in X$. If $B(x_1 : \epsilon) = X$, we stop. otherwise there exists

$x_2 \in X \setminus B(x_1 : \epsilon)$ and thus $d(x_1, x_2) \geq \epsilon$. If $X = B(x_1 : \epsilon) \cup B(x_2 : \epsilon)$, we stop. Otherwise there exists $x_3 \in X \setminus (B(x_1 : \epsilon) \cup B(x_2 : \epsilon))$ and thus $d(x_3, x_1) \geq \epsilon$ and $d(x_3, x_2) \geq \epsilon$. This process must stop after a finite number of steps, for otherwise we obtain a sequence $\{x_n\}$ in X such that

$$d(x_n, x_m) \not< \epsilon \quad \text{if } n \neq m$$

Such a sequence cannot have a convergent subsequence.

Sequentially Compact Metric Space

A metric Space $(X\ d)$ is sequentially compact iff every sequence in X has a convergent subsequence in X

Example # Every finite subset K of a metric space (X, d) is sequentially compact

Sol. # Let $\{x_n\}$ be a sequence of points in K. Since K is finite, at least one point p (say) of K must recurr infinitely often in $\{x_n\}$. Hence $\{p, p, p, \ldots\}$ is a subsequence of $\{x_n\}$ which converges to $p \in K$. Thus K is sequentially compact

Theorem # Metric space $(X\ d)$ is sequentially compact iff it possesses the B.W.P.

Proof # Let $(X\ d)$ be sequentially compact. Let K be an infinite subset of X. We need to prove that K has a limit point. Since K is infinite, we can choose a sequence $\{x_n\}$ of distinct points from K. As X is Seq-Compact, the sequence $\{x_n\}$ must possess a convergent subsequence $\{x_{n_k}\}$. Thus $x_{n_k} \to p$ as $k \to \infty$. This limit implies that each open ball $B(P : \epsilon)$ must contain all

all the points of $\{x_{n_k}\}$ for $k \geq n_0$ (integer). Since all terms in the subsequence $\{x_{n_k}\}$ are distinct, the ball $B(P:\epsilon)$ must contain infinitely many terms of $\{x_n\}$ and therefore of K. This means p is a limit point of K. Thus, every infinite subset K of X has a limit point whence X has the B.W.P.

Converse # Let (X d) possess B.W.P. We need to prove that X is seq-compact. Let K be an infinite subset of X. It has a limit point, since X has B.W.P. Let $\{x_n\}$ be an arbitrary sequence in space X.

(i) If $\{x_n\}$ has finitely many distinct points, then one of the points x_0 occurs infinitely often. Thus $\{x_0, x_0, x_0 \ldots\}$ is a subsequence of $\{x_n\}$ which converges to $x_0 \in X$

(ii) If $\{x_n\}$ has infinitely many distinct points, then range set S of the sequence is infinite. As per Hypothesis. S has a limit point p. Thus given $\epsilon > 0$, the set $S \cap B(P:1/n)$ is infinite ($\epsilon = 1/n$). Now choose $x_{n_1} \in S \cap B(P:1)$, $x_{n_2} \in S \cap B(P:1/2)$; $x_{n_3} \in S \cap B(P:1/3)$ ..... After chosing $x_{n_1}, x_{n_2} \ldots x_{n_k}$, pick $x_{n_{k+1}} \in S \cap B(P, \frac{1}{k+1})$ with $n_{k+1} > n_k$. This subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to p since $x_{n_k} \in S \cap B(P:\frac{1}{k})$ which means $d(x_{n_k}, P) < \frac{1}{k}$

Thus we have extracted a sub-sequence from the sequence $\{x_n\}$ which converges to p. Consequently X must be sequentially compact

Cor Every Compact metric space (X, d) is sequentially Compact.

Proof # Let A be an infinite subset of X, which has no limit point in X. Then for all $a \in A$, there is $r_a > 0$ such that $B(a:r_a) \cap A = \{a\}$ for otherwise

a would be a limit point of A. Obviously

$$\mathcal{G} = \{B(a:r_a) : a \in A\} \cup (X - A) = X$$

$\Rightarrow$ the family $\{B(a:r_a) : a \in A,\ X - A\}$ is an open cover of X and it admits no finite subcover. This means X is not compact which contradicts the hypothesis : X is compact. Thus A must have a limit point in X. This proves that X is sequentially compact [If every infinite set $A \subseteq X$ has a limit point, then $(X.d)$ is sequentially compact B.W.P]

Example # Let A be a sequentially compact subset of a metric space $(X.d)$. Then for each $\delta > 0$, there is a finite number of points $p_1, p_2, \ldots p_n$ in A such that

$$A \subset \bigcup_{j=1}^{n} B(p_j : \delta) \longrightarrow (1)$$

Sol Assume that $A \neq \phi$ and choose a point p_1 in A. Let $\delta > 0$ be given. If $A \subset B(p_1 : \delta)$, then (1) is proved. Otherwise choose $p_2 \in A - B(p_1 : \delta)$. If $A \subset B(p_1 : \delta) \cup B(p_2 : \delta)$, then we stop. Otherwise choose $p_3 \in A \setminus (B_1 \cup B_2)$ where $B_1 = B(p_1 : \delta)$ $B_2 = B(p_2 : \delta)$. We continue this process. If A is covered in a finite no of steps, result (1) is proved. Otherwise there is an infinite sequence $\{p_n\}$ such that p_i is at a distance at least δ from all preceding elements p_j. That is $d(p_i, p_j) \geq \delta$ $\forall\ i \neq j$. This means that, no subsequence $\{p_{n_k}\}$ can converge to a point. This is a contradiction since A is sequentially compact. It follows that selection of $p_1, p_2 \ldots$ must stop after a finite number of steps.

Theorem # (Heine-Borel Theorem): Let $(X\ d)$ be a metric space and $A \subseteq X$ be sequentially compact subset. Prove that A is compact.

Proof # Let $\mathcal{C} = \{G_\alpha : \alpha \in I\}$ be a family of open

sets that covers A. We want to prove that a finite sub-family of C also covers A.

To the contrary, we suppose that no finite sub-family of C covers A and obtain a contradiction

Now by above example for each $\delta = 1/2$, there is a finite number of points $P_{11}, P_{12}, \ldots P_{1m_1}$ in A such that

$$A \subset \bigcup_{j=1}^{m_1} B(P_{1j}; 1/2)$$

Then one of the sets $A \cap \overline{B(P_{1j} : 1/2)}$ $j = 1, 2, \ldots m_1$ is such that an infinite no of open sets of C are need to cover it. Otherwise A would be compact. Let that set $A \cap \overline{B(P_1 : 1/2)} = C_1$ say. Then C_1 is sequentially compact. Again applying above example to C_1 with $\delta = \frac{1}{2^2}$. Then there is a finite no of points $P_{21}, P_{22} \ldots P_{2m_2}$ in C_1 such that

$$C_1 \subseteq \bigcup_{j=1}^{m_2} B(P_{2j}; 1/2^2)$$

one of the balls $B(P_{2j} : 1/2^2)$, $j = 1, 2 \ldots m_2$ is such that an infinite no of open sets of C are needed to cover the portion of A contained in that ball. Label that ball by $\overline{B(P_2 : 1/2^2)}$. We continue in this manner and obtain a sequence of closed balls.

$$B[P_1 : 1/2]\ ;\ B[P_2 : 1/2^2] \ \ldots \ B[P_n : 1/2^n]$$

$\{B_k = B(P_k : 1/2^k)\}$

Such that $P_n \in A \cap B_n$ and an infinite no. of opensets of C are needed to cover $A \cap \bar{B}_n$. Since $d(P_{n-1}, P_n) \leq (\frac{1}{2})^{n-1}$ for each n. The sequence $\{P_n\}$ is cauchy, it must converge in A let $P_n \to P_0 \in A$. Then P_0 lies in some open set G_0 of C. Since $P_n \to P_0$, $\bar{B}_n \subset G_0$ for large value of n. However this contradicts the fact that an infinite no of elements of C are needed to cover $\bar{B}_n \cap A$. This proves that there exists a finite sub-family of C which covers A.

Thus A must be compact.

Lebesgue Number

Let X be a metric space. A real number $\lambda > 0$ is said to be a lebegue number for an open cover C of X if every open ball $B(x:\lambda)$ $\forall x \in X$ is contained in at least one member of C.

Lebesgue Covering Lemma

Let A be a sequentially compact subset of (X,d) and $C = \{G_\alpha : \alpha \in I\}$ be an open cover for A. Then there is a +ve number $\lambda > 0$ (Lebesgue number) such that $\forall a \in A$, the ball $B(a:\lambda)$ is contained in one of the open sets of C.

OR

Every open covering of a sequentially subset of a metric space has a Lebesgue number.

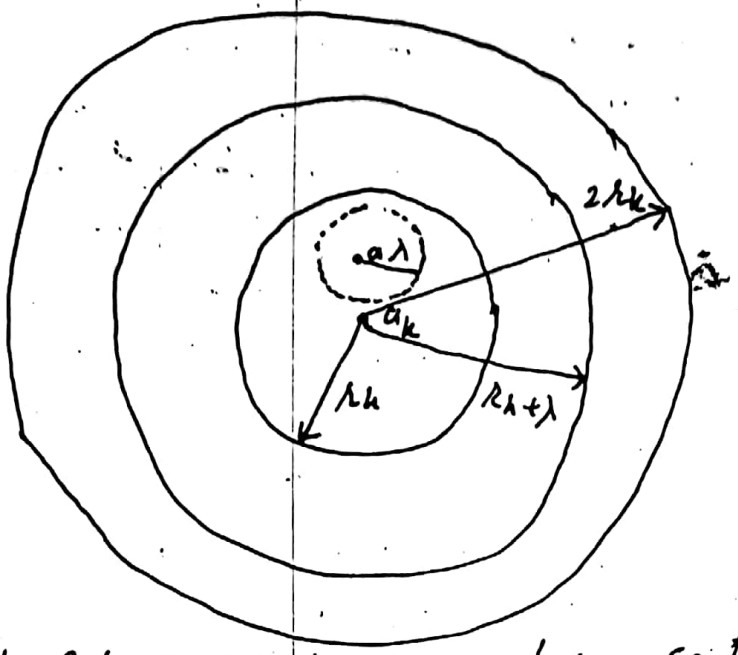
Proof # Since C covers A, each $a \in A$ is a centre of some open balls of radius $2r_a$ say which is entirely in one open set of C. Hence the ball $B(a:r_a)$ is also in the same open set.

let $C_0 = \{B(a:r_a) : \forall a \in A\}$. Then C_0 is an open cover for A. Since A is sequentially compact, by Heine Borel theorem A is compact and open cover $C_0 = \{B(a:r_a) : \forall a \in A\}$ has a finite sub cover $C_0^* = \{B(a_k : r_k) : k = 1, 2, \ldots n\}$.

Let $\lambda = \min\{r_1, r_2, \ldots r_n\}$. Now if $a \in A$, then a is in some ball $B(a_k : r_k)$

Consequently

$$B(a:\lambda) \subset B(a_k : r_k) \subset B(a_k : 2r_k)$$



The ball $B(a_k : 2r_k)$ was chosen so that it lies in a single open set of C. It follows that ball $B(a:\lambda)$ also lies in that single open set of C. Since a is an arbitrary point of A, the situation holds

Note # The sets of an open cover of a compact space overlap each other by a definite amount. This accounts for the existence of Lebesgue numbers.

Different open covers may yield different Lebesgue numbers.

Theorem # Let X be a sequentially compact metric space and C be any open cover for X. Then there is a real number $\epsilon > 0$ such that every open ball of radius ϵ in X is contained in some member of C.

OR

Every open covering of a sequentially compact metric space has a Lebesgue number.

Proof # Suppose C is an open cover of a sequentially compact metric space X and C

has no Lebesgue number. Then for each $n \in N$, there is a point $x_n \in X$ such that $B(x_n : 1/n)$ is not contained in any member of C.

Consider now the sequence $\{x_n\}$ in X. Since X is sequentially compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to x (say) in X.

$\because$ C is an open cover for X

$\therefore$ $x \in U$ for some member U of C

$\because$ U is an open set

$\therefore$ There is an open ball $B(x : 1/p)$, p some natural number such that.

$$x \in B(x ; 1/p) \subseteq U$$

x is the limit point of a subsequence of $\{x_n\}$

Choose a +ve integer k such that

$$d(x_{n_k}, x) < \frac{1}{2p} \quad \text{and} \quad \frac{1}{n_k} < \frac{1}{2p}$$

or $n_k > 2p$

We will show that $B(x_{n_k} \ \frac{1}{n_k}) \subseteq U$

Let $y \in B(x_{n_k} \ \frac{1}{n_k})$

Then $d(y, x_{n_k}) < \frac{1}{n_k}$

Now

$$d(y\ x) \leq d(y\ x_{n_k}) + d(x_{n_k}\ x)$$

$$< \frac{1}{n_k} + \frac{1}{2p}$$

$$< \frac{1}{2p} + \frac{1}{2p} = \frac{1}{p}$$

$$\Rightarrow y \in B(x : \tfrac{1}{p}) \subseteq U.$$

$$\Rightarrow B(x_{n_k} \ \tfrac{1}{n_k}) \subseteq B(x : \tfrac{1}{p}) \subseteq U$$

a Contradiction of the assumption that $B(x_n, \frac{1}{n})$ is not contained in any member of C. Hence there is a real number $\epsilon > 0$ such that any open ball of radius $< \epsilon > 0$ is contained in some member of C.

OR

This can be proved by using Lebesgue covering Lemma as

Let $C = \{G_\alpha : \alpha \in I\}$ be an open cover which has no Lebesgue number. Then $\forall n \in N$ (in fact for all real number $\in > 0$ according to Lebesgue covering Lemma), there is a non-empty subset $A_n \subseteq X$, with $0 < \delta(A) < 1/n$ such that

$$A_n \not\subseteq G_\alpha \quad \forall \alpha \in I \quad \rightarrow (1)$$

For each n choose $x_n \in A_n$, thus getting a sequence $\{x_n\}$ of points of X. Since X is sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$.

Let $x_{n_k} \rightarrow p \in X$, when $k \rightarrow \infty$

Since $p \in X = \bigcup \{G_\alpha : \alpha \in I\}$, so $p \in G_\beta$ for some $\beta \in I$. Now G_β is an open set containing p, so an open ball $B(p:\epsilon)$ exists such that

$$B(p:\epsilon) \subseteq G_\beta.$$

Hence ϵ is the radius of the open ball.

Now $x_{n_k} \rightarrow p$ and $\delta(A_{n_k}) \rightarrow 0$ as $k \rightarrow \infty$.

This implies that there exists a +ve number σ such that

$$d(x_{n_\sigma}\ p) < \epsilon/2, \quad \delta(A_{n_\sigma}) < \epsilon/2$$

Let $y \in A_{n_\sigma}$

$$d(y, p) \leq d(y\ x_{n_\sigma}) + d(x_{n_\sigma}\ p) \leq \delta(A_{n_\sigma}) + d(x_{n_\sigma}\ p)$$

$$[\ x_{n_\sigma} \in A_{n_\sigma}]$$

$$< (\epsilon/2) + (\epsilon/2) = \epsilon$$

$$\Rightarrow d(y\ p) < \epsilon \Rightarrow y \in B(p:\epsilon) \subseteq G_\sigma \Rightarrow y \in G_\sigma$$

we infer that

$$A_{n_\sigma} \subseteq G_\sigma$$

This contradicts (1). Consequently C must possess a Lebesgue number.

Theorem Every sequentially Compact metric space is countable compact.

Proof# Let (X, d) be a sequentially compact metric space. We prove that X is countably compact. If X is finite, then it is trivially countable compact. So let X be infinite and suppose that X is not countably compact. Then there must exist a countable covering $C = \{G_n : n \in N\}$ of X which can not be reduced to a finite sub-covering. We generate a sequence $\{x_n\}$ in X as under.

Let n_1 be the smallest +ve integer such that $X_1 = X \cap G_{n_1} \neq \phi$ and let $x_1 \in X_1$. Let $n_2 > n_1$ be the least +ve integer such that $X_2 = X \cap G_{n_2} \neq \phi$ and let $x_2 \in X_2 \setminus X_1$. Such a point necessarily exists, for otherwise G_{n_1} will cover X. We continue this process to generate a sequence $\{x_n\}$ of points of X such that for any $r \in N$, we have

$$x_r \in X_r \text{ and } x_r \notin \bigcup_{k=1}^{r-1} x_k \qquad [X_r = X \cap G_{n_r}]$$

We now prove that this sequence $\{x_n\}$ has no convergent subsequence in X. Let $p \in X$, then $\exists$ an open set G_p in C such that

$$p \in G_p$$

Since $X \cap G_p \neq \phi$, there exists an integer $r \in N$ such that $G_p = G_{n_r}$. Hence our choice of sequence yields

$$i > r \Rightarrow x_i \notin G_{n_r}$$

This shows that $G_{p_{n_r}}$ is an open set that contains p but does not contain all the points of the sequence from some place on. Also such no subsequence of $\{x_n\}$

This implies that X is not sequentially compact which contradicts the hypothesis. Thus (X, d) is countably compact

Theorem # Every countably compact metric space is sequentially compact

Proof # Let X be countably compact metric space and $\{x_n\}$ be a sequence in X. If the number of terms of $\{x_n\}$ is finite, then a term of is infinitely repeated. Let x_{n_0} be this term. Then $\{x_{n_0}, x_{n_0}, x_{n_0}, \ldots x_{n_0}, \ldots\}$ is a constant subsequence of $\{x_n\}$ and converge to x_{n_0}

If $\{x_n\}$ consists of an infinite no of terms, then as an infinite subset of X, it has a limit point x say (In a countably compact space, every infinite subset has a limit point in X). Since X is a metric space, we can choose a subsequence $\{x_{n_1}, x_{n_2}, \ldots x_{n_m}, \ldots\}$ of $\{x_n\}$ which converges to x in X. Hence X is sequentially compact.

Theorem Every countably compact subset of a metric space (X, d) is sequentially Compact.

Proof # Let A be countably compact subset of X. We prove that A is sequentially compact.

Let $\{x_n\}$ be any sequence in A.

(i) Let $\{x_n\}$ the range of the sequence $\{x_n\}$ be finite, then there exists a point say $x_{n_0} = p$ of $\{x_n\}$ which recurrs infinitely often. This shows that $\{p, p, p, \ldots\}$ is a convergent subsequence of $\{x_n\}$ that converges to p. Consequently A is sequentially compact

(ii) Let the range of the sequence be infinite. Since A is Countably Compact, the range set S must possess a limit point say l_0 in A. This means we can certainly extract a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to l_0$ as $k \to \infty$. This shows that A is sequentially compact.

Problem # Let (X, d) be a metric space, A its compact subset and B, its any other subset. Show that $d(P, B) = d(A, B)$, where $P \in X$

Sol # Since A is compact sequentially compact, it is sequentially compact so that any sequence $\{a_n\}$ in A has a convergent subsequence $\{a_{n_k}\}$. Let

$x_{n_k} \to P \in A$ as $k \to \infty$.

This implies

$$d(a_{n_0}, P) < \delta_k \qquad n \geq n_0 \longrightarrow (1)$$

Let $d(A, B) = \epsilon \geq 0$. We have by definition

$$\epsilon = d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$
$$\leq d(a, b) \longrightarrow (2)$$

(2) shows that there exist point $a_n \in A$, $b_n \in B$ such that

$$\epsilon \leq d(a_n, b_n) < \epsilon + (1/n) \qquad \forall n \in N$$

In particular

$$d(a_{n_0}, b_{n_0}) < \epsilon + (1/n_0) \longrightarrow (3)$$

from (1) & (3)

$$d(P, a_{n_0}) + d(a_{n_0}, b_{n_0}) < \delta_k + \epsilon + (1/n_0) \longrightarrow (4)$$

We want to prove that

$$d(P, B) = d(A, B) = \epsilon.$$

We assume to the contrary. Let $d(P, B) \neq \epsilon$

According to (2)

$d(P\ B) \geq \epsilon$, so we need really show that $d(P,B) > \epsilon$. Thus we can write

$$d(P,B) = \epsilon + \delta \qquad \delta > 0$$

$$\epsilon + \delta = d(P\ B) - \inf\{d(P,b) : b \in B\}$$

$$\Rightarrow \quad \epsilon + \delta \leq d(P, b_n) \qquad \forall\, b_n \in B \qquad \rightarrow (5)$$

In particular $\epsilon + \delta \leq d(P\ b_{n_0}) \quad \forall\, b_{n_0} \in B$

from (4) & (5)

$$d(P, a_{n_0}) + d(a_{n_0}\ b_{n_0}) < d(P, b_{n_0}) \rightarrow (6)$$

which is a rejection of triangle inequality. Hence the assumption. $d(P,B) < \epsilon$ must be rejected and $d(P\ B) = \epsilon \qquad \forall\, P \in X$

(Proved)

Theorem # Let X be a metric space. Then following statements about X are equivalent.

(a) # X is compact

(b) # X is Countably compact

(c) # X is sequentially compact

Proof # (a) $\Rightarrow$ (b)

If X is compact, then every open cover and in particular every countable open cover contains a finite subcover

(b) $\Rightarrow$ (c)

We have proved that every countably compact space is sequentially compact. Hence (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a)

Suppose that X is sequentially compact. Let $\{G_\alpha : \alpha \in I\}$ be an open cover of X

Since X is sequentially compact, every open cover of it contains a Lebesgue number. Thus for ~~every~~ open cover $\{G_\alpha : \alpha \in I\}$, there is a real no

$\epsilon > 0$ such that every open ball of radius ϵ in X is contained in some G_α, $\alpha \in I$.

Also since every sequentially compact space is totally bounded, X is totally bounded and has an ϵ-net for real number $\epsilon > 0$. So for the above $\epsilon > 0$, there is an ϵ-net.

$$F = \{a_1, a_2, \ldots a_k\}$$

such that

$$X = \bigcup_{i=1}^{k} B(a_i, \epsilon)$$

But ϵ is a lebesgue number
Therefore each ball $B(a_i, \epsilon)$ is contained in some member of $\{G_\alpha : \alpha \in I\}$.

Let the members be $G_{\alpha_1}, G_{\alpha_2} \ldots G_{\alpha_k}$.
such that

$$B(a_i; \epsilon) \subseteq G_{\alpha_i} \quad i = 1, 2 \ldots k$$

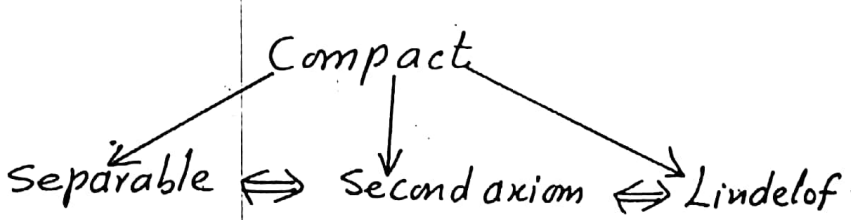
Hence

$$X = \bigcup_{i=1}^{k} B(a_i \epsilon) \subseteq \bigcup_{i=1}^{k} G_{\alpha_i} \subseteq X$$

$$\Rightarrow X = \bigcup_{i=1}^{k} G_{\alpha_i}$$

So X is compact

Comments



Theorem Let X be a metric space, then following statements about X are equivalent

(a) X is compact
(b) X is sequentially compact.
(c) X satisfies the Bolzano, Weirstrass

Proof # We have proved that every compact metric spaces has B.W.P and that a metric space is sequentially iff it satisfies B.W.P.

(a) $\Rightarrow$ (b).

Since every compact space satisfies B.W.P and every space satisfying B.W.P is sequentially compact, therefore

(a) $\Rightarrow$ (b) (b) $\Rightarrow$ (c) is trivial

(c) (b) $\Rightarrow$ (a) Every sequentially compact space is totally bounded and every totally bounded space is compact Thus a space satisfying B.W.P is compact.

Hence (c) $\Rightarrow$ (a)

Theorem Every Countable Compact metric space is 2nd Countable.

Proof # Let X be a Countably compact metric space. Then X is totally bounded because every Countably compact space is totally bounded.

Hence for each natural number k, there is a finite subset F_k such that

$$X = \bigcup_{x \in F_k} B(x\ 1/k) \longrightarrow (1)$$

i.e for every real no $\frac{1}{k}$, there is an ϵ-net.

Let $D = \bigcup_{k=1}^{\infty} F_k$. Then D being the union of Countable no of sets, is Countable. We Shew that D is dense in X. Let $y \in X \setminus D$ and $\epsilon > 0$ be any real number. We can choose a natural number n such that

$$1/n < \epsilon$$

By (1) $y \in B(x, 1/n)$ for some element x of $F_n \subset D$

i.e

$$B(x, 1/n) \cap D \neq \phi$$

$\Rightarrow$ every element in $X \setminus D$ is an adherent point of D. Also every element of D is centre of some open ball which intersects with D. Hence elemets of D^c & D are adherent points of D i.e

$$\overline{D} = X$$

$\Rightarrow$ X Contains a Corentable dense subset of

$\Rightarrow$ X is separable.

Now a ametric space is separable iff it is 2nd Countable

Hence X is 2nd countable space. (Proved)

Theorem # Every Compact metric space is separable.

Proof # Let (X, d) be a Compact metric space. We need to prove that it is a separable (i.e corentable dense subset of X exists)

For a fixed +ve integer n Consider the collection of open balls

$$C = \{ B(x : 1/n) : x \in X \}$$

Then C is an open cover of X. Since X is Compact, Cover C admits a finite Subcover C_0

$$C_0 = \{ B(x_{n_i} : 1/n) : i = 1, 2 \ldots k \}$$

For each $n \in N$, we define finite subset D_n by

$$D_n = \{ x_{n_i} : i = 1, 2 \ldots k \}$$

For each $x \in X$, there is an element x_{n_i} in D_n

such that

$$d(x, x_{n_i}) < \left(\frac{1}{n}\right)$$

because the collection C_0 covers X.

The set $D = \bigcup\{D_n : n \in N\}$ is obviously countable being countable union of countable sets.

We want to prove that D is dense in X i.e $X = \bar{D}$. For this we are to prove that every element of X is an adherent point of D.

Let x be an arbitrary point of X. Suppose G is an open nbd of x. Then there exists an open ball $B(x : 1/n)$ for some $n \in N$ such that

$$x \in B(x : 1/n) \subset G \quad (\because G \text{ is open}) \longrightarrow (1)$$

But, $\forall x \in X$, there exists $x_{n_i} \in D_n \subset D$ such that

$$d(x, x_{n_i}) < (1/n)$$

$$\Rightarrow x_{n_i} \in B(x : 1/n) \subset G \longrightarrow (2)$$

It follows that every nbd of x contains at least one point of D and x must be an adherent point of D

Thus every point $x \in X$ is an adherent point of D which means $X \subseteq \bar{D}$

But always $\bar{D} \subseteq X$

$$\Rightarrow \bar{D} = X$$

$\Rightarrow D$ is a countable dense subset of X.

$\Rightarrow X$ is a separable metric space.

Theorem #(a) Prove that every closed subset of a Countably compact metric space is Countably Compact

(b) :- A Countably compact metric space has the B.W.P

Proof# (a) Let (X, d) be a Countably Compact metric space and F be a closed subset of X. We are to prove that F is Countable Compact

Let $C = \{G_n : n \in N\}$ be any countable open covering of F. Then

$$F \subseteq \bigcup_n G_n$$

$$\therefore X = F \uplus F' \quad \{\uplus \Rightarrow \text{disjoint union.}$$

$$X \subseteq (\bigcup_n G_n) \uplus F'$$

But

$$\bigcup_n G_n \uplus F' \subseteq X$$

$$\Rightarrow X = (\bigcup_n G_n) \uplus F' \longrightarrow (1)$$

$\Rightarrow$ Collection $\{C, F'\}$ is a Countable open covering of X. Since X is Countably Compact and F' does not cover any part of F, it follows that there exists a finite open covering $C_0 = \{G_{n_i} : i = 1, 2, \ldots m\}$ of C such that

$$F \subseteq \bigcup_{i=1}^{m} G_{n_i}$$

$\Rightarrow$ F is Countably Compact

(b) # Let (X, d) be a Countably Compact

and A be an infinite subset of of X.
We prove that A is a limit point (B.W.P)
Assume that A has no limit point.
Since every infinite set contains a denumberable set, there exists a set

$$B = \{ t_n : n \in N \}$$ such that $B \subseteq A$

Evidently B has no limit point since $A \supseteq B$ has no limit point. Thus B is closed (it has all it limit points vacuously)

Obviously, no t_n is a limit point of B. So for each t_n in B, there exists an open set G_n (ball) containing t_n but not containing any other point of B except possibly t_n. Thus. $C = \{ G_n : n \in N \}$ is a countable open covering of B and as such

$$B \subseteq \cup_n G_n$$

$$\Rightarrow X = B \cup B' \subseteq (\cup_n G_n) \cup F'$$

Thus the collection $\{C, F'\}$ is a countable open covering of X

However this open covering has no finite sub-covering. For if we remove a single set G_n, from this collection, then t_n will not be contained in the union of the remaining members This means that (X, d) is not countably compact.

With this contradiction, we reject the assumption ; A has no limit point. Thus every infinite. subset of a countably compact metric space has a limit point (B.W.P)

ϵ-Net

Let (X, d) be a metric space. A finite subset F of X is said to an ϵ-net for X iff given any point $p \in X$, there exists a point q of F such that $d(p, q) < \epsilon$

Totally Bounded OR Pre-Compact Set

A set A in a metric space (X, d) is called pre-compact or totally bounded if for each $\epsilon > 0$, there is a finite set of points say $x_1, x_2 \dots x_n$ in X such that A is contained in the union of open spheres $S(x_i, \epsilon)$ $i = 1, 2 \dots n$ i.e

$$A \subseteq \bigcup_{i=1}^{n} S(x_i : \epsilon)$$

Note #(1) Obviously, (X, d) itself is totally bounded iff, for every $\epsilon > 0$, X has an ϵ-net.

(2) The empty set ϕ is clearly bounded

(3) Evidently, the definition of total boundedness can be re written as follows.

A subset A of a metric space (X, d) is pre-compact if corresponding to any $\epsilon > 0$, there exists a finite number of sets $A_1, A_2, \dots A_n$ each having diameter not exceeding ϵ such that

$$A \subseteq A_1 \cup A_2 \cup \dots \cup A_n.$$

Lemma # Let (X, d) be a Countably Compact metric space, then X has an ϵ-net for every $\epsilon > 0$ i.e X is totally bounded.

Proof # If possible, let there be an $\epsilon > 0$ s.t X has no ϵ-net. for $p_1 \in X$, obviously there exists at least one point p_2 such that

$d(p_1, p_2) \geq \epsilon$ Since the subset of X consisting of the single point p_1 is not an ϵ-net for X. Since $\{p_1, p_2\}$ is not an ϵ-net for X, there must exist a point $p_3 \in X$ which is not in $U_1 \cup U_2$

where $U_1 = S(P_1, \epsilon)$ and $U_2 = S(P_2, \epsilon)$
Also $d(P_i, P_j) \geq \epsilon$ for $i \neq j$, $i = 1, 2, \ldots, 3$, $j = 1, 2, 3 \ldots$
We continue this process. Suppose we found n points $P_1, P_2 \ldots P_n$ in X such that $d(P_i P_j) \geq \epsilon$ for $i \neq j$ $i = 1, 2, \ldots n$, $j = 1, 2 \ldots n$ and define $U_i = S(P_i, \epsilon)$ for $i = 1, 2 \ldots n$. Let $F = \{P_1, P_2 \ldots P_n\}$, then F is not an ϵ-net for X enabling the existence of point P_{n+1} not in $\bigcup_{i=1}^{n} U_i$; at once, we thus have

$$d(P_i, P_j) \geq \epsilon \quad i \neq j \quad i = 1, 2 \ldots (n+1), \; j = 1, 2 \ldots (n+1)$$

By mathematical induction, we now have a sequence $\{P_n\}$ of distinct points of the countably compact metric space (X, d) such that

$$d(P_i P_j) \geq \epsilon \quad i \neq j$$

If $K = \bigcup_{n=1}^{\infty} \{P_n\}$, K is an infinite set in the countably compact space X and so has a limit point (B.W.P) q in X. Thus $S(q : \epsilon/3)$ must contain infinitely many elements of K contradicting the fact that $d(P_i P_j) \geq$ $i \neq j$. Thus X must have an ϵ-net.

Theorem (Be Explained) Every Countably compact metric space (X, d) is separable

Proof # Since every countably compact metric space has an ϵ-net for each $\epsilon > 0$, therefor every $n \in N$, X must have an $\frac{1}{n}$ net F_n.

Let ~~$F = \{P_n\}$ $n \in N$~~

Let $F = \bigcup_{\infty} F_n$, then F being countable union of finite sets, is countable. We shall that

$$X = \bar{F}$$

, thereby proving separability

Let $p \in X$. For an arbitrary real number $\delta > 0$, let $m \in N$ be such that $\frac{1}{m} < \delta$. Since F_m is an $\frac{1}{m}$-net, there exists a point $f \in F_m$ such that $d(p, f) < \frac{1}{m} < \delta$. Since $f \in F$, $f = p_k$ for some $k \in N$. We have thus proved that for every $\delta > 0$, $S(p, \delta)$ contains a point of F. Hence $\bar{F} = X$

Thus X is separable.

Relatively Compact # Set

A set A in (X, d) is called relatively compact if $\bar{A}$ is compact in X

Theorem # A set A in a metric space is relatively compact iff every sequence in A has a convergent subsequence. (Easy to prove)

Theorem # A relatively compact set A in metric space (X, d) is totally bounded.

Proof # If possible, Let A be not totally bounded, then there exists an $\epsilon > 0$ such that A is not contained in the union of any finite no of open spheres each with radius ϵ. For x_1 in A, choose x_2 in A such that $d(x_1, x_2) \geq \epsilon$. Then choose $x_3 \in A$ so that $d(x_1, x_3) \geq \epsilon$ and $d(x_2, x_3) \geq \epsilon$. When $x_1, x_2 \ldots x_n$ have been chosen, there is some point x_{n+1} of A not in the set $\bigcup_{i=1}^{n} S(x_i, \epsilon)$. We have thus chosen a a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points in A. Since A is realtively compact, sequence $\{x_n\}$ has a convergent subsequence and thus the set $F = \bigcup_{n=1}^{\infty} \{x_n\}$ has a limit point say x. Hence $S(x, \epsilon/3)$ must contain infinitely many elements of $\{x_n\}$ contracting

the definition of $\{x_n\}$. Hence A must be totally bounded.

Note Converse of this theorem is not true

Cauchy Sequence

A sequence $\{x_n\}$ in a metric space X is called Cauchy sequence if every pe assigned $\epsilon > 0$, there exists a natural no n_0 such that

$$d(x_m, x_n) < \epsilon \quad \text{whenever } n \geq n_0, m \geq n_0$$

Complete Metric Space

A metric space (X.d) is called complete metric space if every Cauchy sequence is convergent in metric space.

Note # Every convergent sequence is a Cauchy sequence.

Theorem # If (X.d) is a Complete metric space, a totally bounded set A in X is relatively Compact.

Proof # We knows that a set is relatively compact if every sequence in A has a convergent subsequence in A. Therefore it is sufficient to shew that every sequence in A has a Cauchy subsequence

Let (x_n) be any sequence in A. Since A is totally bounded, A is contained in the finite union of open spheres each of radius 1 ($\because$ A has an ϵ-net for every $\epsilon > 0$). At least one of these spheres must contain a subsequence from $\{x_n\}$. Let such a subsequence be denoted by $x_{11}, x_{12}, x_{13} \ldots$

Next if we take $\epsilon = 1/2$, by property of total boundedness of A, a finite union of spheres of radius $1/2$ would contain A. Find a sphere of radius $1/2$ which contains a subsequence of $\{x_{1n} : n = 1, 2 \ldots\}$, say

$\{x_{2n} : n = 1, 2, \ldots\}$. We proceed by induction obtaining sequences $\{x_{ki} : i = 1, 2, \ldots\}$ $(k = 1, 2, \ldots)$ each sequence a subsequence of the predecessor and the kth sequence is contained in a sphere of radius $\frac{1}{k}$.

Thus if $m > k$, $d(x_{mm}, x_{kk}) < \frac{1}{m}$ and thus $d(x_{mm}, x_{kk}) \to 0$ as $m \to \infty$ and $k \to \infty$ giving $\{x_{nn} : n = 1, 2, \ldots\}$ to be a Cauchy subsequence of $\{x_n\}$

Note from theorem it is clear that if a complete space (X, d) is totally bounded then it must be compact.

Lemma # A Cauchy sequence is convergent iff it has a convergent subsequence.

Proof # If a Cauchy sequence is itself convergent, then it is a convergent subsequence of itself.

Conversely, let a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) have a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Then there exists a $x \in X$ such that, corresponding to a preassigned $\epsilon > 0$ there is $p_1 \in N$ such that

$$d(x_{n_k}, x) < \epsilon/2 \qquad k > p_1$$

Since $\{x_n\}$ is a Cauchy sequence, there exists $p_2 \in N$ such that

$$d(x_n, x_m) < \epsilon/2 \qquad m, n > p_2$$

Choose k sufficiently large and greater than p_1 such that $n_k > p_2$, then

$$d(x_{n_k}, x_m) < \epsilon/2 \qquad m > p_2$$

Then $d(x_m, x) \leq d(x_m, x_{n_k}) + d(x_{n_k}, k) < \epsilon/2 + \epsilon/2 = \epsilon$ whenever $m > p_2$

Hence sequence $\{x_n\}$ is convergent.

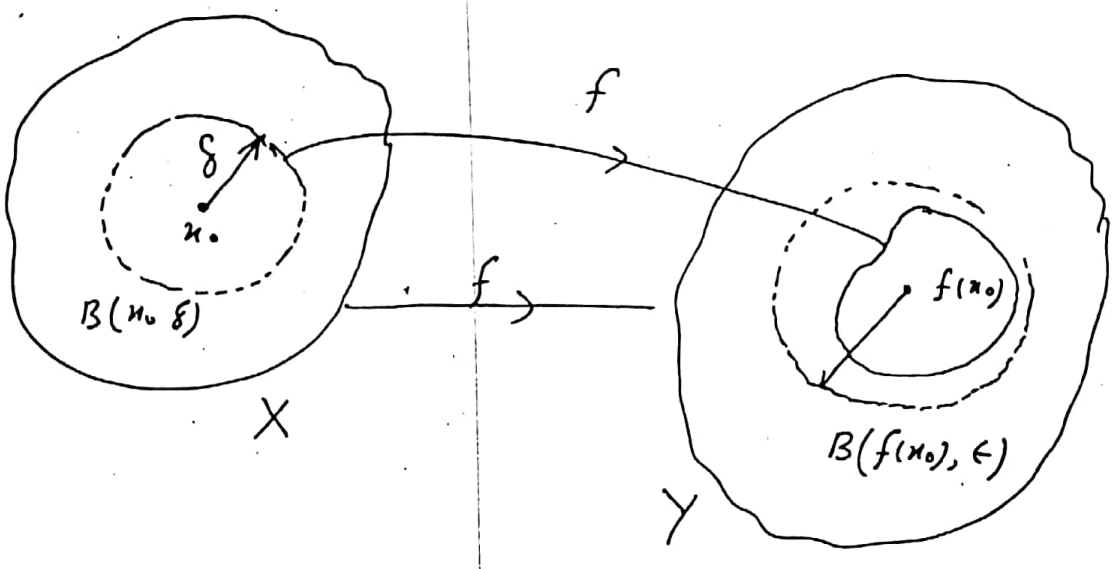
Theorem # A Compact metric space is complete

Proof # Let $\{x_n\}$ be a Cauchy sequence in (X, d). Since X is sequentially compact, $\{x_n\}$ has a convergent subsequence and hence is convergent. Thus (X, d) is compete

Continuous function on Metric Space

Let (X, d_1) and (Y, d_2) be two metric spaces and $f: X \to Y$ be a function. Then f is said to be Continous at a point $x_0 \in X$ iff for each open ball $B_Y(f(x_0), \epsilon)$, centred at $f(x_0)$, there exists an open ball $B_X(x_0, \delta)$, centred at x_0, such that

$$f\left(B_X(x_0, \delta)\right) \subseteq B_Y(f(x_0); \epsilon)$$



This definition amounts to

f is continous at $x_0 \in X$ iff for each $\epsilon > 0$, there exists a $\delta > 0$ (δ depends upon both x_0 & ϵ) such that

if $d_1(x_0, x) < \delta$, then $d_2(f(x_0), f(x)) < \epsilon \quad \forall x \in X$

i.e

$$d_1(x_0, x) < \delta \Rightarrow d_2(f(x_0), f(x)) < \epsilon \quad \forall x \in X$$

Note If $X = Y = R$ and $d_1 = d_2$ is the usual metric for R, then this definition is exactly the definition of continuity for real valued functions defined on a subset of R.

Uniform Continuity

A mapping f, of a metric space (X, d_1) into a metric space (Y, d_2) is uniformly continuous on X iff for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that

$$d_2(f(x), f(y)) < \epsilon \quad \text{, whenever } d(x, y) < \delta$$

Comments # (1) For continuous function, given $\epsilon > 0$, δ depends on both ϵ and point x_0 at which continuity is being considered. but for uniformly continuous function given $\epsilon > 0$, δ depends on ϵ but not on point

(2) A uniform continuous function is necessarily continuous, however the converse is not necessarily true.

Uniform Isomorphism

If the function f is 1-1 onto, f and f^{-1} are both uniformly continuous, then f is called a uniform isomorphism

Definition # Let (X, d_1) and (Y, d_2) be metric spaces and $f: A \to Y$, where $A \subset X$, f is said to be uniformly continuous on A provided that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if x_1, x_2 are any two points of A with $d_1(x_1, x_2) < \delta$, then $d_2(f(x_1), f(x_2)) < \epsilon$

Compactness and Continuity

Theorem # If f is a continuous function from a compact metric space (X, d_1) into a metric space (Y, d_2), then $f(X)$ is compact

OR

Continuous image of compact metric space is compact

Proof # Let $f(x_n)$ be a sequence in $f(X)$. Then $\{x_n\}$ is a sequence in X and since X is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to a point a in X (A metric space is compact iff every sequence in it has a convergent subsequence). Since f is continuous at a, $\lim_{k \to \infty} f(x_{n_k}) = f(a)$.

Thus $f(X)$ is compact

Theorem # Let f be a continuous mapping of a compact metric space (X, d_1) into a metric space Y. Then f is uniformly continuous.

Proof # Since f is continuous, it is continuous at each point of X. Thus for each $x \in X$, there a $\delta(\epsilon, x) > 0$ or $\delta_x(\epsilon)$ for given $\epsilon > 0$ such that

$$\forall x' \in X,\ d_1(x, x') < \delta \Rightarrow d_2(f(x), f(x')) < \epsilon/2 \quad \rightarrow (1)$$

The collection $\{B(x, \frac{\delta_x(\epsilon)}{2}) : x \in X\}$ forms an open covering for compact space X. Thus there exists a finite subcovering $\{B(x_i, \frac{\delta_{x_i}(\epsilon)}{2}) : i = 1, 2 \dots n\}$ or $\delta_{x_i} = \delta_i$

Let $\delta = \min\{\delta_1/2, \frac{\delta_2}{2} \dots \frac{\delta_n}{2}\}$. Then $\delta > 0$

Now let x, x' be any two points of X such that

$$d(x, x') < \delta$$

Since $\{B(x_i, \frac{\delta_{x_i}(\epsilon)}{2}) : i = 1, 2 \ldots n\}$ is an open cover for X, x must belong to some open ball say $B(x_j, \frac{\delta_{x_j}}{2}) = B(x_j, \frac{\delta_j}{2})$

$$\Rightarrow \quad d_1(x, x_j) < \delta_j/2 < \delta_j = \delta_{x_j}$$

Also

$$\begin{aligned} d_1(x_j, x') &\leq d(x_j, x) + d(x, x') \\ &< \delta_j/2 + \delta \\ &< \delta_j/2 + \frac{\delta_j}{2} = \delta_j \quad \because \delta = \min\{\tfrac{\delta_1}{2}, \tfrac{\delta_2}{2} \ldots \tfrac{\delta_n}{2}\} \\ &= \delta_{x_j} \end{aligned}$$

from ①

$$d_1(x, x_j) < \delta_{x_j} = \delta_j \Rightarrow d_2(f(x), f(x_j) < \epsilon/2$$

$$d_1(x_j, x') < \delta_{x_j} = \delta_j \Rightarrow d_2(f(x_j), f(x') < \epsilon/2$$

Hence

$$\forall\, x, x' \in X,\ d_1(x, x') < \delta \Rightarrow d_2(f(x), f(x')$$

$$< d_2(f(x), f(x_j)) + d_2(f(x_j), f(x'))$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

Thus for given $\epsilon > 0$ and for any two points $x, x' \in X$ we have found δ such that

$$d_1(x, x') < \delta \Rightarrow d_2(f(x), f(x') < \epsilon$$

Hence f is uniformly continuous function.

II Method

Let $\epsilon > 0$. Since f is continuous on X, for each point $x \in X$, there is a $\delta_x = \delta_x(\epsilon) > 0$ such that

$\forall x' \in X, d_1(x,x') < \delta_x \Rightarrow d_2(f(x), f(x') < \epsilon/2 \rightarrow (1)$

The collection $\{B(x, \delta_x) : x \in X\}$ is an open cover of X.

Since X is compact, X has a Lebesgue number λ, say, for this open cover. Thus each open ball of radius less than λ is contained in some open ball $B(x, \delta_x)$

Now let for any $u, v \in X$

$$d_1(u, v) < \lambda$$

Then if $A = \{u, v\}$, then $\delta(A) < \lambda$ so that u, v are in a $B(x, \delta_x)$. so

$d(x, u) < \delta_x$, $d(x, v) < \delta_x$

from (1), we have

$d_2(f(u), f(x)) < \epsilon/2$ & $d_2(f(x), f(v) < \epsilon/2$

Hence

$\forall u, v \in X$, $d_1(u, v) < \lambda$ (lambda) $\Rightarrow d_2(f(u), f(v))$

$\leq d_2(f(u), f(x) + d_2(f(x), f(v)$

$< \epsilon/2 + \epsilon/2 = \epsilon$

Hence f is uniformly continuous.

Corollary # If f is a continuous real-valued function on a closed and bounded subset X of R^n, f is uniformly continuous.

Proof # $\because$ A subset X of R^n is compact iff X is closed and bounded.

$\therefore$ X is compact being closed and bounded subset of R^n

By above Theorem any continuous function from X into R (metric space with usual absolute metric) is uniformly continuous

Theorem # Prove that a continuous image of a sequentially compact metric space is sequentially compact

Proof # Let (X, d_1) be sequentially compact metric space and (Y, d_2) be any other metric space. Let $f: X \to Y$ be a continuous mapping we are to prove that $f(X)$ is sequentially compact Let $\{y_n\}$ be any sequence in the image space $f(X)$. The for each +ve integer n, there exists a point x_n in X such that $f(x_n) = y_n$ This generates a sequence $\{x_n\}$ in X.

Since X is sequentially compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$, as $k \to \infty$ Using continuity of f we obtain

$$x_{n_k} \to p \Rightarrow f(x_{n_k}) \to f(p) \in f(X) \quad [\because p \in X]$$

Thus $f(x_{n_k})$ i.e y_{n_k} is a subsequence of $\{y_n\}$ that converges to $f(p)$ in $f(X)$. This implies that $f(X)$ is sequentially compact.